HENRY SHARP, JR.

FINITE FUNCTIONS

An Introduction to Combinatorial Mathematics

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HENRY SHARP, JR.

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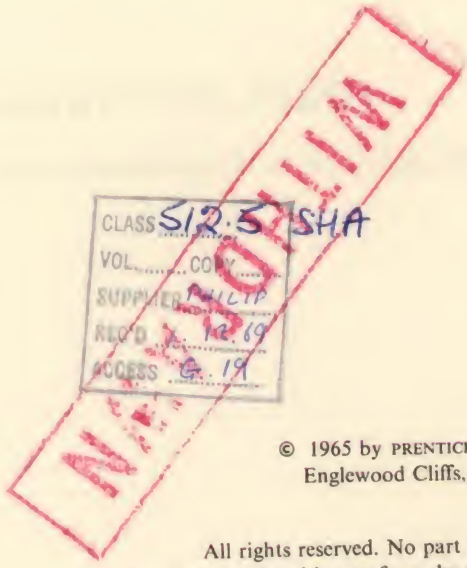

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Preface

The *function* concept lies at the heart of all mathematics, and upon it (to borrow a phrase of James R. Newman) mathematicians have "lavished exquisite care." Among various possibilities, the "ordered pair" definition of function now seems to be that most widely used. Nevertheless, many who meet this idea for the first time are wary of it—perhaps because a comfortably vague notion must be supplanted by a clean and economical statement which admits no deviation from its intent.

The first half of this volume has as its major purpose the exploration of the function concept in a finite setting, where familiarity can be gained by an explicit listing of all (or most) of the ordered pairs. Because the ideas may be novel to many readers, examples and illustrations are profuse. Although a text cannot replace the guidance of a good teacher, it can encourage independent thought about the subject it treats. To this end, exercise lists are included with each chapter in which the intent is not so much to provide drill in techniques as to promote understanding. Answers or hints are supplied for nearly all exercises, and suggestions for further study are incorporated in the Reference List and Key at the end of the book.

The study of functions on finite sets leads inevitably to questions of a combinatorial nature. The definition of *combination* and *permutation* in terms of such functions is quite natural, and serves

as a modern introduction to some of the long-known counting formulas indispensable in the elementary theory of probability. Chapter Five is the combinatorial bridge linking the function concept with the enumeration problems to which the second half of this volume is devoted. It is *not* intended that this second half be viewed as a compartmentalized course in combinations and permutations. It *is* intended that the study of this material reinforce the student's general training in mathematics by demonstrating the application of certain fundamental tools and concepts in the attractive setting of elementary combinatorics.

Within each chapter, major definitions and theorems are numbered and examples are identified by capital letters. To facilitate reference or casual reading, the symbol "‖" is used to indicate the end of each proof. Another notational device requires comment. The author believes that "$\binom{n}{r}$," widely used in representing binomial coefficients, is one of the most awkward symbols currently appearing in the literature of elementary mathematics. Because of this prejudice, "$\binom{n}{r}$" is replaced by "$\{n\}_r$," which closely parallels the symbol "$(n)_r$" now accepted as a notation for the number of permutations of r things out of n. Nothing, of course, prevents the reader who fails to share this prejudice from ignoring the suggested deviation and replacing the replacement.

The present volume is planned so as to be reasonably self-contained, and most of the material has been classroom tested. The intended audience includes: (1) secondary school teachers in institutes or other special programs, (2) senior high school or first-year college classes, and (3) teachers of elementary mathematics, for use in reference or self-study.

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FINITE FUNCTIONS:

An Introduction to Combinatorial Mathematics

Chapter One

SETS

We shall consider the word "set" as used in this study to be undefined; that is, we shall make no attempt to reduce this idea to one more primitive than itself. By an examination of the properties and behavior attributed to sets we hope to bring the concept clearly into focus. The classification of objects with respect to a given set is subject to two ground rules:

the objects in the set are distinct from one another; and

it is possible (theoretically, at least) to distinguish those objects in the set from those not in the set.

In the study of set theory the nature of the objects in a set is immaterial, and this lends the subject an abstractness which in turn produces a surprisingly wide range of applicability.

An object in a set is called an *element of* (or *in*) the set.

An element of a set is said to *belong to* the set.

A set is said to *contain* each element in the set.

Capital letters are used frequently to indicate sets, lower-case letters are used frequently to indicate elements.

To indicate that an element designated by the symbol "a" is in the set designated by the symbol "M" we use "$a \in M$." We use

"$a \notin M$" to mean that a does not belong to M. The symbol "$\in$" is the *membership* symbol indicating that the element named belongs to (or is a *member of*) the set named.

A set may be exhibited as a list of symbols representing the elements which belong to the set. It is customary to do this by enclosing the list in braces. For example, $\{1, 2, 3\}$.

A set may be designated by a verbal description. For example, $\{1, 2, 3\}$ can be described as the set of positive integers less than 4.

A verbal description may be cast into mathematical form by enclosing in braces a symbol representing an arbitrary element of the set, followed by some characterizing property or properties. For example, the set above may be indicated

$$\{x : x \text{ an integer and } 1 \leqq x \leqq 3\}.$$

Although the elements in a set must be distinct from one another, this does not imply that the elements in any concrete application must be distinguishable from one another.

A. If we receive change at a bank for a dollar bill we might be given a half-dollar, a quarter, two dimes, and a nickel, all in mint condition. There is no doubt that we have a set consisting of five coins, yet it may be beyond our power to detect any difference between the two dimes.

B. $\{a, e, i, o, u\}$ is the set of vowels in the English alphabet.

$\{\alpha, \beta\}$ is the set of Greek letters from which the word "alphabet" is derived.

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is the set of digits.

$\{., ?, !\}$ is a set of terminal punctuation marks.

1 DEFINITION: *Two sets M and N are said to be* ***different*** *iff there exists an element which belongs to one set but not the other.*

We use *iff* as an abbreviation for *if and only if.*

Let M and N be sets. If M and N are different we write $M \neq N$; otherwise we write $M = N$, and say that M and N are *equal.*

2 DEFINITION: *Let M and N be sets. The set M is said to be a* ***subset*** *of N, or to be* ***contained in*** *N, iff each element of M is also an element of N.*

If M is contained in N then we say also that N *contains* M, and we denote this idea symbolically by $M \subset N$ or by $N \supset M$. The symbol "$\subset$" is the *inclusion* symbol indicating that one set is contained in (or is included as a subset of) another.

The set theoretic distinction between inclusion and membership is quite reasonable, but in some ways is quite subtle. Let M be a set. Since each element of M is also an element of M (this sounds like lunacy, but is only redundancy!), Definition 2 tells us that M is a subset of M. We may write, therefore, $M \subset M$, and it is no surprise that

any set is a subset of itself.

But the situation as regards membership is completely different. Since the nature of the objects in a set is immaterial, it is perfectly reasonable to consider a set the elements of which are themselves sets. We are led, then, to wonder whether a set may be considered a member of itself. In the early days of set theory it was believed reasonable to assume the existence of a set M for which it is true that $M \in M$. It soon appeared, however, that this assumption implied a number of unsuspected contradictions; to avoid them, we deny the assumption and agree that

no set is a member of itself.

3 THEOREM: *Let M and N be sets. Then $M = N$ iff both $M \subset N$ and $N \subset M$.*

Proof: We must prove two statements: (1) if $M = N$, then $M \subset N$ and $N \subset M$; and (2) if both $M \subset N$ and $N \subset M$, then $M = N$. We consider each statement separately.

(1) If $x \in M$ but $x \notin N$ then by Definition 1 the hypothesis $M = N$ is contradicted. Thus $M \subset N$. Similarly, $N \subset M$.

(2) If $M \neq N$, then there is an element in one set (say M) that is not in the other. Thus M is not contained in N, which contradicts the hypothesis that each set is contained in the other. Since the assumption that $M \neq N$ leads to a contradiction, it follows that $M = N$. ‖

4 DEFINITION: *Let M and N be sets. We say that M is a* ***proper subset*** *of N iff both $M \subset N$ and $M \neq N$.*

In this study we shall not make use of a notation which distinguishes proper subsets. In the few instances when it is of importance in an argument that one set be a proper subset of another, we shall so indicate verbally.

The set concept in itself implies nothing with regard either to the number of elements in a set or to the order in which the elements might chance to be listed. Referring to the definition of set equality, $\{a, e, i, o, u\}$ is the same as $\{e, a, u, o, i\}$ since no element belongs to one set but not the other.

C. A positive integer is called a *prime number* (or a *prime*) iff the set of its factors contains exactly two elements. The numbers 2 and 11 are prime, since $\{1, 2\}$ is the set of factors of 2 and $\{1, 11\}$ is the set of factors of 11. The numbers 1 and 12 are not prime, since the set of factors of 1 is $\{1\}$ and the set of factors of 12 is $\{1, 2, 3, 4, 6, 12\}$ and neither set contains exactly two elements. Note that every prime number except 2 is an odd number, for each even number greater than 2 has at least three factors. Thus the set of all even prime numbers is $\{2\}$.

Sets consisting of one and only one element are called *singleton* sets (from the familiar bridge-table terminology).

Set theory demands that a distinction be made between a singleton set $\{s\}$ and the single element s that belongs to it. Although $s \in \{s\}$, it is not the case that $s = \{s\}$.

Continuing the bridge-table analogy, consider a bridge hand consisting of four spades, five hearts, and four diamonds. Such a hand is usually described as void in clubs. We might, therefore, be tempted to assert that the "set" of clubs in this hand is void. The problem is: is this an acceptable description of a set? Neither of the ground rules stated at the beginning of this section is violated by the admission of such a set; on the other hand, no logical necessity demands its admission. The important point to remember is that the choice made is not a consequence of the theory but is based upon convenience.

We agree to admit the existence of a set containing no element, called the *void set*, or the *empty set*, or the *null set*, and we denote this set by the symbol $\varnothing$. We agree further that if A is any set, then $\varnothing \subset A$.

D. The set of all even prime factors of 15 is $\varnothing$.

E. Let $\{p, q, r\}$ be a set. We may list each subset of this set systematically as follows, according to the number of elements it contains:

0-element subset: $\varnothing$,

1-element subsets: $\{p\}$, $\{q\}$, $\{r\}$,

2-element subsets: $\{p, q\}$, $\{p, r\}$, $\{q, r\}$,

3-element subset: $\{p, q, r\}$.

5 DEFINITION: *Given a set M. The set of all subsets of M is called the* ***power set*** *of M.*

For reasons which will appear in a later section, the power set of M is denoted by 2^M. Thus if $A = \{p, q, r\}$, then $2^A = \{\varnothing, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$.

F. The number of elements contained in $\varnothing$ is zero; however, the number of elements contained in $\{\varnothing\}$ is one. This distinction is important: the symbol "$\{\varnothing\}$" represents the singleton set having as its only element the void set. Thus $\{\varnothing\}$ is the power set of $\varnothing$. Because of our agreement that $\varnothing$ is a subset of any set, it happens in this case not only that $\varnothing \in \{\varnothing\}$ but also that $\varnothing \subset \{\varnothing\}$. Notice that it is *not* the case either that $\varnothing \in \varnothing$ or that $\varnothing = \{\varnothing\}$.

In some cases it may be impractical to try to list all of the elements in a given set. If sets of this kind are to be suggested by a list of elements, it is customary to use the device indicated in the following two examples.

G. The set of all positive integers is denoted by $\{1, 2, 3, \ldots\}$ and the set of all nonnegative integers is denoted by $\{0, 1, 2, \ldots\}$.

H. The set $\{1, 2, 3, 4, 5\}$ is called an *initial segment of the positive integers*, or in short an *initial segment*. Initial segments will occur with great frequency in the later sections, and we shall consistently use an abbreviation such as $\{1, 2, 3, 4, 5\} = I_5$, where the subscript on I indicates the number of elements in the initial segment. In general, if k is any positive integer, then the set $\{1, 2, 3, \ldots, k\}$ is an initial segment denoted by I_k.

EXERCISES

1. Given the set $M = \{x, y\}$.

(a) Exhibit the four subsets of M.

(b) Exhibit two subsets, neither of which is contained in the other.

(c) Exhibit the distinct proper subsets of M.

(d) If any one of the following statements is incorrect, explain why:

(1) $\varnothing \subset M$, (2) $\varnothing \in M$,

(3) $\{\varnothing\} \subset M$, (4) $x \in M$,

(5) $\{y\} \subset M$, (6) $\{y\} \in M$.

2. Given the set $M = \{p, q, r, s\}$.

(a) Exhibit the 2-element subsets, each of which contains s.

(b) Exhibit the 2-element subsets, none of which contains s.

(c) How many subsets are there of each size: 0-element, 1-element, 2-element, 3-element, and 4-element?

(d) If any one of the following statements is incorrect, explain why:

(1) $\{p, q\} = \{q, p\}$, (2) $\{r\} = \{s\}$,

(3) $\{p\} = p$, (4) $q \subset \{q\}$,

(5) $\{r\} \subset \{r\}$, (6) $\{q, s\} \subset \{p, q, r\}$,

(7) $s \in M$, (8) $\varnothing \subset \{s\}$,

(9) $\varnothing \in p$, (10) $\{p\} \neq \{r, s\}$.

3. Exhibit as a set the 15 smallest primes.

4. Exhibit as a set the real-number solutions of each quadratic equation:

(a) $x^2 - 1 = 0$,

(b) $x^2 = 0$,

(c) $x^2 + 1 = 0$.

5. How many elements are there in each of the following sets?

(a) $\varnothing$, (b) $\{\varnothing\}$,

(c) $\{\varnothing, \{\varnothing\}\}$, (d) $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

6. For the number 1617

(a) exhibit the set of prime factors,

(b) exhibit the set of factors.

7. (a) Exhibit the power set 2^A where $A = \{x: x \text{ a prime factor of } 1155\}$.

(b) Is there a relation between the set of factors of 1155 and the set 2^A?

8. (a) How many numbers are there in the initial segment I_{20}, but not in I_{13}?

(b) If $n \geqq k > 0$, how many numbers are in the initial segment I_n, but not in I_k?

Chapter Two

NEW SETS FROM OLD

In any investigation to which set theory is applied there is imposed a limitation on the kinds of elements to be tolerated. For example, the volume of a right circular cylindrical container of radius r and height h may be calculated by the formula $V = \pi r^2 h$. But in order that the calculation make sense physically, r and h must be replaced by positive real numbers. Hence, in this situation the set of elements to be tolerated as replacements for r and h is $\{x: x \text{ real and } x > 0\}$.

In the theory of sets we formalize the limitation required above by asserting that every set entering into the discussion is in fact a subset of some given set called (suggestively) the *universe*. We denote this universal set by the letter "S."

Let M be a subset of S and let x represent an element in S. Either $x \in M$ or $x \notin M$. We can, therefore, classify all elements of S into two sets: one is M, the other is the set of all elements in S which fail to belong to M.

1 DEFINITION: *Assume that $M \subset S$. The set N is called the* ***complement*** *of M with respect to S iff*

$$N = \{x: x \text{ belongs to } S \text{ but } x \text{ does not belong to } M\}.$$

We denote the complement of M with respect to S by $S - M$. The "minus sign" which we have just borrowed from arithmetic is given an even wider set theoretic usage.

2 DEFINITION: *Let M and N be sets (remember they are assumed to be subsets of S). The set theoretic* ***difference*** *$M - N$ is the set*

$$\{x: x \text{ belongs to } M \text{ but } x \text{ does not belong to } N\}.$$

Note that

$$N - M = \{x: x \text{ belongs to } N \text{ but } x \text{ does not belong to } M\}.$$

A. Let $B = \{x: x \text{ a prime}\} = \{2, 3, 5, 7, 11, \ldots\}$. Then

$I_5 - B = \{1, 4\}$,

$B - I_5 = \{x: x \text{ a prime greater than } 5\} = \{7, 11, 13, \ldots\}$.

Also, $I_5 - \varnothing = I_5$, $\varnothing - I_5 = \varnothing$,

$I_5 - \{5\} = I_4$, $\{5\} - I_5 = \varnothing$,

$I_5 - \{0\} = I_5$, $\{0\} - I_5 = \{0\}$.

B. In this example let the universe S be the set of all positive integers. The complement of the initial segment I_5 is

$$S - I_5 = \{6, 7, 8, \ldots\} = \{x: x \text{ an integer and } x > 5\}.$$

$S - I_7 = \{8, 9, 10, \ldots\}$, and

$S - (S - I_7) = S - \{8, 9, 10, \ldots\} = I_7$.

Note that $I_5 \subset I_7$ and that $(S - I_7) \subset (S - I_5)$ since

$$\{8, 9, 10, \ldots\} \subset \{6, 7, 8, \ldots\}.$$

The preceding example suggests two general results which are stated below as theorems.

3 THEOREM: *If $M \subset S$ then $S - (S - M) = M$.*

Proof: Let $p \in S - (S - M)$. Then p is not in the complement of M. But every element of S is in either M or its complement, hence $p \in M$. By Definition 2, page 2, $(S - (S - M)) \subset M$. Now let $p \in M$. Then p is not in the complement of M, hence it is in the

complement of the complement of M. As above $M \subset (S - (S - M))$. By Theorem 3, page 3, $(S - (S - M)) = M$. ‖

4 THEOREM: *Let M and N be subsets of S. If $M \subset N$ then $(S - N) \subset (S - M)$.*

Proof: Let $p \in S - N$. Then p does not belong to N, and by hypothesis $p \notin M$. Thus $p \in (S - M)$. Hence $(S - N) \subset (S - M)$. ‖

5 DEFINITION: *Let M and N be sets. The set A is called the* ***union*** *of M and N iff $A = \{x: x \in M \text{ or } x \in N\}$.*

The union of M and N is denoted by $M \cup N$. Note that union is associative, $(L \cup M) \cup N = L \cup (M \cup N) = L \cup M \cup N$, and commutative, $M \cup N = N \cup M$. The word "or" in Definition 5 is inclusive: either one or the other or both.

6 DEFINITION: *Let M and N be sets. The set A is called the* ***intersection*** *of M and N iff $A = \{x: x \in M \text{ and } x \in N\}$.*

The intersection of M and N is denoted by $M \cap N$. Note that intersection is commutative and associative.

C. In this example let the universe be the set of digits,

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Let $M = \{0, 1, 2, 4, 8\}$, and let $N = \{0, 3, 5, 7\}$. Then
$M \cap N = \{0\}$ (do not confuse this with $\varnothing$),
$M \cup N = \{0, 1, 2, 3, 4, 5, 7, 8\}$ (note that one of the ground rules forbids that 0 be listed twice), and

$$(S - N) - (S - M) = \{1, 2, 4, 8\}.$$

In the early literature of set theory the union of two sets was often called the sum of the sets and the intersection of two sets was often called the product of the sets. Making use of these operations, mathematicians have constructed an extensive theory called the "algebra of sets," which in some ways is surprisingly like the more familiar algebra of numbers but in other ways is strikingly

different. For the purposes of this study only a reasonable facility in handling these operations is necessary. The following example gives a hint of the general algebraic properties of sets, but we shall not pursue the matter further.

D. Let S, M, and N be as given in Example C. Then

$$M \cup M = M, \qquad M \cap M = M,$$

$$M \cup (S - M) = S, \quad M \cap (S - M) = \varnothing,$$

$$(M \cap N) \subset M \subset (M \cup N), \qquad (M \cap N) \subset N \subset (M \cup N),$$

$$(S - M) \cup (S - N) = S - (M \cap N) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

$$(S - M) \cap (S - N) = S - (M \cup N) = \{6, 9\}.$$

A device which is sometimes helpful in illuminating these concepts was described by the English logician John Venn in 1880; it is now called the *Venn diagram*. Let the universe S be represented by the points enclosed in a rectangle, and let an arbitrary subset M of S be represented by the points enclosed in a circle drawn inside the rectangle (as in Figure 2.1). Note that $S - M$ is that part of the rectangle not in the hatched area. Two subsets M and N of S may be represented as in Figure 2.2. In the figure, M is denoted by the horizontally hatched area, N by the vertically hatched area, $M \cap N$ by the crosshatched area, and $M \cup N$ by the hatched area. In Figure 2.3 the set theoretic difference is indicated. The horizontally hatched area represents the difference $M - N$, while the vertically hatched area represents the difference $N - M$. As is made clear in the diagram, if $M \neq N$, then the two set theoretic differences are different.

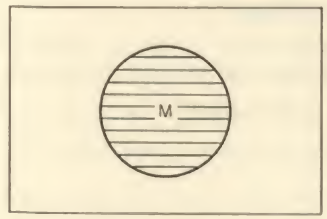

FIGURE 2.1

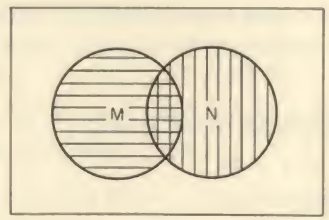

FIGURE 2.2

E. Out of a set of 100 students, 25 take mathematics, 10 take both mathematics and English, and 32 take neither mathematics nor English. To find how many of these students take English but not mathematics, we might use the Venn diagram in Figure 2.4. Let the horizontally hatched area represent the mathematics students, M, and let the vertically hatched area represent the English students, E. The rectangle must include 100 students. There are 10 students in $M \cap E$, hence there are 15 in $M - E$. There are 32 in $S - (M \cup E)$, hence in $E - M$ there are $100 - (32 + 15 + 10) = 43$.

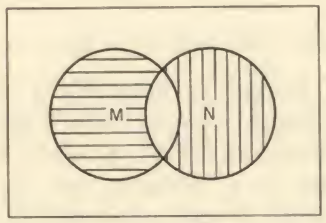

FIGURE 2.3

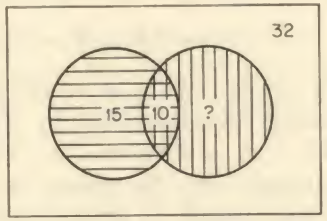

FIGURE 2.4

EXERCISES

1. Let S be the set of all prime numbers.
   (a) Is there an element $x \in S$ such that $\{x, x + 1\} \subset S$?
   (b) Name at least six elements $x \in S$ such that $\{x, x + 2\} \subset S$.
   (c) Is there an element $x \in S$ such that $\{x, x + 1, x + 2\} \subset S$?
   (d) Describe the set $S - \{2\}$.
   (e) How many elements are there in the sets:
   (1) $I_2 - S$, (2) $I_3 - S$,
   (3) $I_{20} - S$, (4) $I_{50} - S$?

2. Describe the set $I_n - I_m$ if
   (a) $n > m$,
   (b) $n = m$,
   (c) $n < m$.

3. Let S be the set of all prime numbers and let M be the set of digits. Exhibit each of the following sets.
   (a) $S \cap M$, (b) $M - S$,
   (c) $M - (S \cap M)$, (d) $(S - M) \cap (M - S)$.
   (e) If any one of the following statements is incorrect, explain why:
   (1) $0 \in M$, (2) $0 \in (M - S)$,
   (3) $\{0\} = \varnothing$, (4) $\{0, 1\} \subset (M - S)$,
   (5) $(3 + 8) \in (S \cup M)$, (6) $(3 + 9) \in (S \cup M)$,
   (7) $(S - I_2) \subset (S - I_3)$, (8) $(S - M) \subset (S - I_7)$.

4. Show that $I_{10} - I_3 = I_{10} \cap (I_{20} - I_3)$.

5. Let M be the set of digits.
   (a) Exhibit the set $M - I_9$.
   (b) Show that $M = I_9 \cup \{0\}$.
   (c) Show that $I_9 - M \neq M - I_9$.
   (d) Show that $I_{10} - M = I_{10} - I_9$.
   (e) Show that $I_{12} \cup M \neq I_{12}$.
   (f) Show that $I_{12} \cap M = I_9$.

6. Let L, M, N represent subsets of a universe S. Sketch Venn diagrams for each of the following:
   (a) $L \cap (S - (M \cup N))$,
   (b) $(M \cap N) \cap (S - L)$,
   (c) $(L \cap (S - M)) \cup ((S - L) \cap M)$,
   (d) $((L \cup M) \cap N) \cup (L \cap M)$,
   (e) $S - (L \cap M \cap N)$.

7. If M and N are subsets of a universe S, show that
$$M - N = M \cap (S - N).$$

8. Referring to Figure 2.3, the *symmetric difference* of the sets M and N is (by definition) $(M - N) \cup (N - M)$.
   (a) Exhibit the symmetric difference of the set I_{10} and the set of digits.
   (b) Exhibit the symmetric difference of the sets I_3 and I_7.

9. Of a group of 120 students, 90 take English and 72 take history. If 10 students take neither, how many take both?

10. In a certain city there are three major newspapers (A, B, C), of which at least two are read by 35 per cent of the population. It is known that newspaper C is read by 45 per cent of the population, that newspapers A and B are read by 15 per cent, and that all three are read by 10 per cent. What percentage of the population read only newspaper C?

Chapter Three

CARTESIAN PRODUCTS

By symbols such as "(x, y)," "(a, b)," "$(1, \sqrt{3})$," we shall designate *ordered pairs.* This concept is critical in all our future work, and we shall rely on the following discussion to crystallize the idea.

Let two mutually perpendicular lines be drawn in the plane, one horizontal and the other vertical. The point of intersection is called the *origin of coordinates* and is the zero point on each line. The lines are called *coordinate axes.* On each axis we mark an arbitrary unit interval, and this establishes on each axis a correspondence between the set of real numbers and the points on the line. From the origin, points on the horizontal axis to the right and on the vertical axis upward correspond to the positive real numbers, and oppositely to the negative real numbers. This establishes a *rectangular coordinate system* in the plane.

Now consider any two real numbers given in a definite order, for example the numbers 1 and $\sqrt{3}$. Find the point on the horizontal axis which corresponds to the first, and let V be the line through that point parallel to the vertical axis. Find the point on the vertical axis which corresponds to the second, and let H be the line through that point parallel to the horizontal axis. Let P be the point of intersection of V and H, and assign to P the "address" $(1, \sqrt{3})$,

called an ordered pair. Thus every pair of real numbers in a given order corresponds to a point in the plane, and these numbers are called the *rectangular coordinates* of the point. The first number is called the *1st coordinate* (or the *abscissa*), and the second number is called the *2nd coordinate* (or the *ordinate*). The axes of a coordinate system separate the remaining points of the plane into four sets which correspond to the four possible combinations of signs on the coordinates: upper right (called the *first quadrant*), both coordinates positive; upper left (called the *second quadrant*), 1st coordinate negative, 2nd coordinate positive; lower left (called the *third quadrant*), both coordinates negative; lower right (called the *fourth quadrant*), 1st coordinate positive, 2nd coordinate negative. If either coordinate is zero, then the point lies on one of the axes; if both coordinates are zero, then the point is the origin of coordinates.

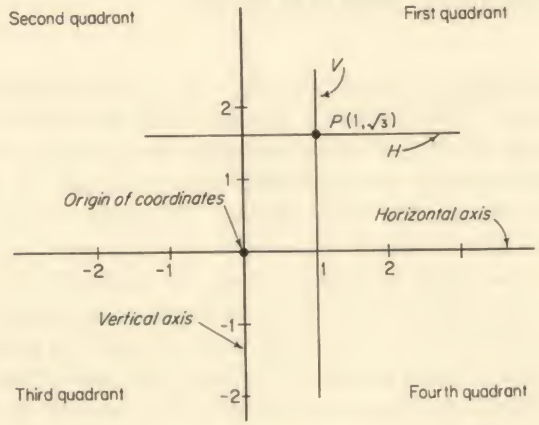

FIGURE 3.1

1 DEFINITION: *The ordered pairs* (a, b) *and* (c, d) *are* ***equal****, denoted by* $(a, b) = (c, d)$*, iff both* $a = c$ *and* $b = d$.

With this definition it is possible to talk about sets of ordered pairs, an ordered pair being a *single element* in such a set. Since

it is possible to tell whether ordered pairs are equal, it is possible also to determine whether sets of ordered pairs are equal.

A. A point P in a rectangular coordinate system is a *lattice point* iff each coordinate of P is an integer. (For simplicity we identify a point by its coordinates, as in Figure 3.2.) The *unit circle* is the set $\{(x, y): x \text{ and } y \text{ real and } x^2 + y^2 = 1\}$. The unit circle contains a subset of four lattice points,

$$\{(1, 0), (0, 1), (-1, 0), (0, -1)\}.$$

The set of lattice points inside the unit circle is $\{(0, 0)\}$.

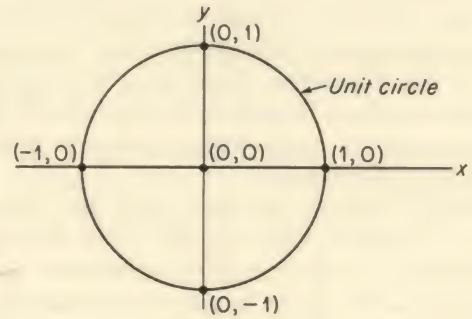

FIGURE 3.2

B. Let S be the set of all lattice points, let A be the set of lattice points with positive 1st coordinate, and let B be the set of lattice points with positive 2nd coordinate. Then $A \cap B$ denotes the set of all lattice points in the first quadrant, and $S - (A \cup B)$ denotes the set of lattice points either in the third quadrant or on a nonpositive part of a coordinate axis.

2 DEFINITION: *Let* M *and* N *be sets. A set is called the* ***cartesian product*** *of* M *and* N*, denoted by* $M \times N$*, iff*

$$M \times N = \{(x, y): x \in M \text{ and } y \in N\}.$$

C. If R represents the set of all real numbers, then the set of all ordered pairs representing points in the plane is the cartesian

product $R \times R$. If N represents the set of all integers, then the set of all lattice points is the cartesian product $N \times N$. If I represents the set of all positive integers, then the set of all lattice points in the first quadrant is the cartesian product $I \times I$.

The cartesian product $I_3 \times I_2$ is the set

$$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}.$$

The cartesian product $I_2 \times I_3$ is the set

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

If either M or N is empty, then $M \times N$ is empty.

The preceding discussion in this chapter is fundamental in the study of plane analytic geometry, and the name "cartesian product" commemorates the pioneering work in analytic geometry done by the seventeenth-century French mathematician René Descartes. In fact, a plane with a rectangular coordinate system is often referred to as a *cartesian plane.*

The problems of analytic geometry arise from efforts to study the properties of certain subsets of the plane. For example, the familiar equation $y = x^2$ simply concentrates our attention on certain points in the plane, namely those such as $(1, 1)$, $(0, 0)$, $(\sqrt{2}, 2)$, $(-2, 4)$, in which the 2nd coordinate is the square of the 1st coordinate. Thus the equation defines a certain subset of $R \times R$.

In a similar way, given an arbitrary cartesian product, our attention is directed usually at some subset, and for this reason it is desirable to have certain descriptive terminology available. In fact, much of what is known as mathematical analysis can be cast in the language of ordered pairs and consists in the study of special kinds of subsets of a cartesian product.

Since a cartesian product $M \times N$ is the set of ordered pairs $\{(x, y)\colon x \in M \text{ and } y \in N\}$, any subset of $M \times N$ also is a set of ordered pairs. In mathematics the word "relation" is used in designating a set of ordered pairs. Thus any subset of $M \times N$ is called a *relation in* $M \times N$. There are, of course, many such relations; in fact, the set of all relations in $M \times N$ is simply the power set of $M \times N$. Among these relations, the smallest is $\varnothing$ and the largest is $M \times N$ in the sense that if Z is any relation in $M \times N$ then $\varnothing \subset Z$ and $Z \subset (M \times N)$.

3 DEFINITION: *Let Z be a relation in $M \times N$. A subset D of M is called the* ***domain*** *of Z iff*

$$D = \{x\colon x \text{ is a 1st coordinate of some ordered pair in } Z\}.$$

A subset R of N is called the ***range*** *of Z iff*

$$R = \{x\colon x \text{ is a 2nd coordinate of some ordered pair in } Z\}.$$

D. Consider the cartesian product $I_3 \times I_2$. Let Z be the relation $\{(1, 1), (1, 2), (2, 1), (3, 1)\}$. Then the domain of Z is I_3 and the range of Z is I_2.

Let Z' be the relation $\{(1, 2), (2, 2), (3, 2)\}$. Then the domain of Z' is I_3 and the range of Z' is $\{2\}$.

Let Z'' be the relation $\{(3, 1), (3, 2)\}$. Then the domain of Z'' is $\{3\}$ and the range of Z'' is I_2.

E. Let I be the set of positive integers. As noted earlier, $I \times I$ is the set of all lattice points in the first quadrant of the cartesian plane. Let Z be the following relation in $I \times I$:

$$Z = \{(x, y)\colon x \text{ an even positive integer and } y \text{ an odd positive integer less than } x\}.$$

This relation is pictured as the marked intersections in Figure 3.3. The domain of Z is the set of all even positive integers, and the range of Z is the set of all odd positive integers.

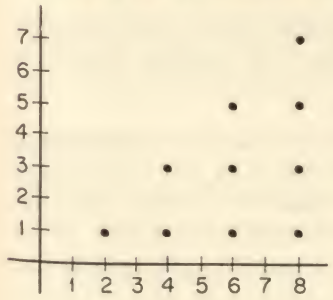

FIGURE 3.3

Given a set M, relations in $M \times M$ having M as domain play an important role in mathematics. They are called relations *on M into itself.* In the event that the range of the relation also is M, we say that the relation is *on M onto itself.* Referring to Example D, since $I_2 \subset I_3$ each one of the relations Z and Z' is on I_3 *into* itself, but neither one is on I_3 *onto* itself. The relation $\{(1, 1), (1, 2), (2, 1), (3, 3)\}$ is on I_3 onto itself, since the

domain is $\{1, 2, 3\}$ and the range also is $\{1, 2, 3\}$. The reader is urged to sketch this relation, as well as those described in Example D, in the manner suggested in Figure 3.3.

We shall illustrate another sometimes helpful method of representing a relation by considering certain subsets of $M \times M$, where $M = \{a, b, c, d, e\}$. First construct a 5-by-5 block of squares, labeling the rows and columns by the elements in M as in Figure 3.4. Given a relation Z, if $(x, y) \in Z$ then we enter a "1" in the square corresponding to the "x" row and "y" column; we enter a "0" in all other squares. The table of *zeros* and *ones* thus produced is sometimes called the *relation matrix* corresponding to Z.

F. Let $M = \{a, b, c, d, e\}$ and let Z be the relation consisting of those ordered pairs in which the 1st coordinate does not follow the 2nd coordinate in alphabetic order. The relation matrix for $Z = \{(a, a), (a, b), \ldots, (b, b), (b, c), \ldots, (e, e)\}$ is shown in Figure 3.5.

	a	b	c	d	e
a					
b					
c					
d					
e					

FIGURE 3.4

	a	b	c	d	e
a	1	1	1	1	1
b	0	1	1	1	1
c	0	0	1	1	1
d	0	0	0	1	1
e	0	0	0	0	1

FIGURE 3.5

	a	b	c	d	e
a	0	1	1	1	1
b	1	0	1	1	1
c	1	1	0	1	1
d	1	1	1	0	1
e	1	1	1	1	0

FIGURE 3.6

Let Z' be the relation consisting of those ordered pairs in which the 1st and 2nd coordinates differ. The relation matrix is shown in Figure 3.6.

Let Z be a relation on M into itself (both relations in Example F happen to be onto, but this is not excluded by use of the word "into"). If for each $x \in M$, $(x, x) \in Z$ then Z is called *reflexive*. If for each $x \in M$, $(x, x) \notin Z$ then Z is called *irreflexive*. If for each $(x, y) \in Z$ it is the case that $(y, x) \in Z$, then Z is called *symmetric*. If for each $(t, x) \in Z$ and $(x, y) \in Z$ it is the case that $(t, y) \in Z$, then Z is called *transitive*.

In a relation matrix the diagonal from upper left to lower right is called the *main diagonal*. Returning to Example F,

Z is reflexive: each main diagonal entry is 1,
Z' is irreflexive: each main diagonal entry is 0,
Z' is symmetric: the table is unchanged by rotation about the main diagonal,
Z is not symmetric: e.g., $(a, b) \in Z$ but $(b, a) \notin Z$,
Z is transitive,
Z' is not transitive: e.g., (a, b) and $(b, a) \in Z$ but $(a, a) \notin Z$.

Relations, of course, may be neither reflexive nor irreflexive. For example, the relation $\{(x, a): x \in M\}$ is neither.

In a more extended survey of relation theory it would be possible to establish a relatively simple criterion for transitivity, but the tabular condition is not so transparent as those related to reflexivity, irreflexivity, and symmetry.

Let Z be a relation on the set M onto itself. The set M is said to be *partially ordered* by the relation Z provided Z is transitive, reflexive, and for no elements $x \neq y$ in M is it the case that both $(x, y) \in Z$ and $(y, x) \in Z$. The latter condition, together with reflexivity, is sometimes called *antisymmetry* and may be described as follows: if $(x, y) \in Z$ and $(y, x) \in Z$, then $x = y$.

Let Z be a relation on the set M onto itself. The relation Z is called an *equivalence relation* provided Z is transitive, reflexive, and symmetric.

Each of the special relations mentioned above is a valuable and frequently used tool in modern mathematical analysis. Enumeration of relations of various kinds is an important problem in the branch of mathematics called *combinatorial analysis*. We shall be concerned in later chapters with certain simple problems of this sort.

EXERCISES

1. Sketch in the cartesian plane several elements in the set of lines:
   (a) $\{y = mx: m \text{ any real number}\}$,
   (b) $\{y = mx + 1: m \text{ any negative real number}\}$,
   (c) $\{y = 2x - b: b \text{ any positive real number}\}$,
   (d) $\{\alpha x + \beta y = 1: \alpha \text{ and } \beta \text{ any positive real numbers}\}$.

2. Given the set of parabolas in the cartesian plane $\{y = 2x^2 + a: a \text{ any real number}\}$. By sketching several elements in this set discover an algebraic distinction between those members in the set for which (1) $a < 0$, (2) $a = 0$, (3) $a > 0$.

3. In this exercise, draw sketches if necessary.†
   (a) In the cartesian plane, how many lattice points
      (1) are less than 1 unit from the origin,
      (2) are less than 2 units from the origin,
      (3) are less than 3 units from the origin?
   (b) Investigate the truth of the following assertion for k a positive integer less than 8: If $N(k)$ represents the number of lattice points less than k units from the origin, and if $A(k)$ represents the area of a circle of radius k units, then $N(k) < A(k) < N(k + 1)$.

4. Let $A = \{0, 1, 2\}$ and let $B = \{2, 3, 4, 5, 6\}$.
   (a) How many elements are there in $A \times B$?
   (b) If each element $(x, y) \in A \times B$ is written as a fraction $\frac{x}{y}$, how many different numbers are represented?
   (c) Exhibit as in Figure 3.3 the relation in $A \times B$ having elements (x, y) corresponding to numbers $\frac{x}{y}$ greater than $\frac{1}{3}$. What is the domain, and what is the range, of this relation?

5. In the cartesian product $I_6 \times I_6$ define a relation
   $$Z = \{(x, y): x \text{ is a factor of } y\}.$$
   (a) Exhibit the relation pictorially in two ways: as in Figure 3.3 and as in Figure 3.4.
   (b) Is this relation
      (1) reflexive,
      (2) symmetric,
      (3) transitive?

† For a discussion related to this problem, see D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (New York: Chelsea Publishing Co., 1952), pp. 32–35.

6. A relation Z in $M \times N$ is called *trivial* if $Z = \varnothing$ or if $Z = M \times N$. Let $M = \{a, e, i, o, u\}$. Construct the relation matrix for a nontrivial relation on M into M
   (a) which is onto,
   (b) which is reflexive,
   (c) which is irreflexive,
   (d) which is symmetric,
   (e) which is transitive,
   (f) which partially orders M,
   (g) which is an equivalence relation.

7. Construct the relation matrices for each relation on I_2 into itself.

8. Construct the relation matrix for each reflexive, symmetric relation on I_3 into itself.

9. Attempt some systematic enumeration of the relations on I_3 onto itself. (This exercise is intended to illustrate the complexity of the general problem. Do not devote a great deal of time to it.)

10. Consider the relation matrix of a relation Z in the cartesian product $I_n \times I_n$. Prove that if Z is both reflexive and symmetric, then there are at least two rows in the matrix having the same sum. (This is the "acquaintance" problem: If n people are gathered in a room, then at least two have the same number of acquaintances in the room.)

Chapter Four

FUNCTIONS

Among all of the relations in a given cartesian product $M \times N$, many are characterized by conditions which are so relaxed that no very useful theory about them has been built up. By imposing certain restrictive conditions, however, we may produce various classes of relations that have proved to be of fundamental importance in the development of mathematics.

1 DEFINITION: *Let Z be a relation in $M \times N$. Z is said to be* ***on*** *M* ***into*** *N iff the domain of Z is M. Z is said to be* ***on*** *M* ***onto*** *N iff the domain of Z is M and the range of Z is N.*

A. Let M be the set of even positive integers and let I be the set of all positive integers. The relation

$$Z = \{(x, y): x \in M \text{ and } y \text{ an odd positive integer less than } x\}$$

is on M into I. In this relation there is one ordered pair in which the number "2" is the 1st coordinate, there are two ordered pairs in which the number "4" is the 1st coordinate, and in general there are n ordered pairs in which the number "$2n$" is the 1st coordinate. The relation

$$Z' = \{(x, y): x \in M \text{ and } y = x/2\}$$

is on M onto I. This relation is "onto" because each positive integer occurs as a 2nd coordinate in some ordered pair: if p is any positive integer then $2p$ belongs to M and the ordered pair $(2p, p)$ is in the relation Z'.

The condition "on M onto N" is a special case of "on M into N." Every relation which is on M onto N is also on M into N. In Example A, Z' is on M into I as well as being on M onto I; Z on the other hand is not on M onto I, for the range of Z includes no even integer.

The two relations in Example A illustrate another important distinction. Notice that in Z there may be many ordered pairs in which the same 1st coordinate appears (for example, there are three ordered pairs having the number "6" as 1st coordinate: (6, 1), (6, 3), (6, 5)); while in Z' there is exactly one ordered pair having a given 1st coordinate (for example, the only ordered pair having the number "6" as 1st coordinate is (6, 3)).

2 DEFINITION: *A relation on M into N is called a* ***function*** *iff no two different ordered pairs in the relation have the same 1st coordinate.*

It is the usual practice in mathematics to denote functions by letters such as f, g, F, G, ϕ, or other. To indicate that a function f is on M into N we use the symbol $f: M \to N$. This means that the domain of f is M and that the range of f is contained in N.

Note that a function is a special case of a relation. Every function is also a relation, but not every relation is a function. In Example A, the relation Z' is also a function, but the relation Z is not a function.

If f denotes a given function on M into N, then we use the symbol "$f(a)$" to represent the particular element in N that is the 2nd coordinate in the ordered pair having a as its 1st coordinate. Thus if a is in the domain of f, then $(a, f(a))$ is an ordered pair in the function f. The element $f(a)$ in N is called the *functional value* corresponding to a. Note that if f is not *onto* N then there is at least one element in N which is not a functional value; on the other hand, if f is onto N then each element in N is a functional value. The set of all functional values in N is sometimes denoted by $f(M)$, and this subset of N is simply the range of f. If $f(M) = N$ then f is on M onto N.

B. Each of the following sets of ordered pairs is a relation in $A \times A$ where A is the set $\{0, 1, 2, 3\}$; but note that Z is a function while Z' is not.

$$Z = \{(0, 2), (1, 1), (2, 2), (3, 3)\},$$

$$Z' = \{(0, 0), (0, 2), (1, 1), (1, 2), (1, 3), (2, 2), (3, 1), (3, 3)\}.$$

Note that Z is on A into A but not *onto* A, Z' is on A onto A, and that $Z \subset Z'$.

Replacing Z by f (to coincide with the terminology in the preceding paragraph), we note that $f(0) = 2$, $f(A) = \{1, 2, 3\}$, $f(A)$ is a proper subset of A, and that 0 is not a functional value.

3 DEFINITION: *A* ***variable*** *on a set M is a symbol (or name) which may be replaced by any element in the set M. If a given function f has domain D and range R, then a variable on D is called an* ***independent variable*** *for f and a variable on R is called a* ***dependent variable*** *for f.*

C. The formula $2x + 1$ may be used in a natural way to define a function. For example, let the domain be $\{0, 1, 2, 3\}$ and let x be the independent variable. If we replace x by 0 in the formula we calculate the corresponding number $2 \cdot 0 + 1 = 1$, which we call $f(0)$ and we agree that the ordered pair $(0, 1)$ belongs to the function. In a similar way, we obtain the ordered pairs $(1, 3)$, $(2, 5)$, $(3, 7)$. This set of ordered pairs is called the function on $\{0, 1, 2, 3\}$ defined by the formula $2x + 1$. The range of f is $\{1, 3, 5, 7\}$. If $A = \{0, 1, 2, 3\}$ and if B is any set containing $\{1, 3, 5, 7\}$, then f is a function on A into B.

Note that the procedure followed in Example C shows that a function f may be defined by a formula provided the domain is assigned.

The function f may be considered a relation in any cartesian product $A \times B$ such that A is the domain and B contains $f(A)$. In particular, f may be considered a relation in $A \times f(A)$. We shall often use the symbol "$f(x)$" to denote a formula, which by the method suggested in Example C, can be used to define a function f on some given domain.

D. Let A be the set $\{0, 1, 2, \ldots\}$, and consider the functions defined on the domain A by the formulas:

$$f(x) = x^2,$$

$$g(x) = 2^x,$$

$$F(x) = x,$$

$$G(x) = 1.$$

Explicitly these functions are:

$$f = \{(0, 0), (1, 1), (2, 4), \ldots\},$$

$$g = \{(0, 1), (1, 2), (2, 4), \ldots\},$$

$$F = \{(0, 0), (1, 1), (2, 2), \ldots\},$$

$$G = \{(0, 1), (1, 1), (2, 1), \ldots\}.$$

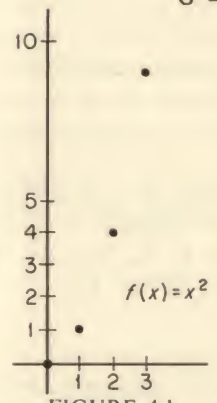

FIGURE 4.1

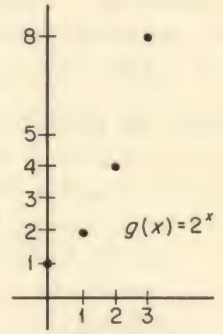

FIGURE 4.2

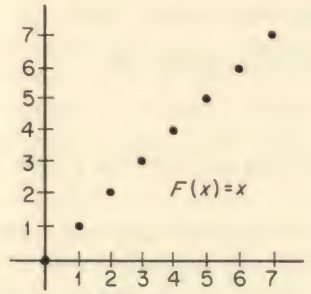

FIGURE 4.3

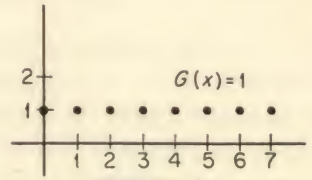

FIGURE 4.4

The fact that a function by definition is a single-valued relation (that is, there is one and only one element in the range corresponding to any element in the domain) does not in any way imply a restriction on the number of elements in the domain that may be related to a given element in the range. An extreme example is the *constant function*, in which the range is a singleton set. The function G in Example D is a constant function; the range of G is $\{1\}$.

Consider now a cartesian product $M \times N$. Although the class of all functions on M into N is severely restricted when compared with the class of all relations on M into N, even this class is too extensive for some purposes. In the next definition we shall consider a restriction designed to eliminate duplication of range elements in f. Recall that $f: M \to N$ denotes a function f such that the domain is M and $f(M) \subset N$.

4 DEFINITION: *Consider the function $f: M \to N$. Then f is called a* ***one-to-one function*** *on M into N iff each two of the ordered pairs in f differ both in 1st coordinates and in 2nd coordinates.*

Among the functions described in Example D, f, g, and F are one-to-one, but G is not one-to-one. The function F is called the identity function on A. In general, let M be any set. Then the *identity function* on M is the function $\{(x, x): x \in M\}$.

Every identity function is one-to-one.

5 DEFINITION: *Let M be any set and let $\mathscr{P}$ represent a property which may or may not be possessed by the elements of M individually. Let $k: M \to \{0, 1\}$ be a function such that*

$$k(x) = \begin{cases} 1 & \text{if } x \text{ has the property } \mathscr{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Then k is called a ***characteristic function on M for the property $\mathscr{P}$.***

In a later chapter we shall be concerned with characteristic functions in general, and we shall call any function $k: M \to \{0, 1\}$ a *characteristic function*. In Example D the function G may be considered a characteristic function: in particular, a characteristic function for the property: "... is a member of A."

E. Let a, b, c, d, e represent individual human beings, and let $M = \{a, b, c, d, e\}$. Let $\mathscr{P}$ be the property: "possesses blue eyes." If, for example, only b and c have blue eyes, then the characteristic function on M for $\mathscr{P}$ is

$$\{(a, 0), (b, 1), (c, 1), (d, 0), (e, 0)\}.$$

F. Consider the set $I_9 = \{1, 2, \ldots, 9\}$ and let $\mathscr{P}$ be the property: "is an odd prime." Then

$$k = \{(1, 0), (2, 0), (3, 1), (4, 0), (5, 1), (6, 0), (7, 1), (8, 0), (9, 0)\}.$$

In the following example we illustrate a function defined by a technique not met before in this study. These function values will occur frequently in later chapters.

G. Let $A = \{0, 1, 2, \ldots\}$, the set of nonnegative integers. Corresponding to each element in A, we calculate the functional value by the formula

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ x \cdot f(x - 1) & \text{if } x \geqq 1. \end{cases}$$

The first few function values are found as follows: $f(0) = 1$ (by definition), $f(1) = 1 \cdot f(0) = 1 \cdot 1 = 1$, $f(2) = 2 \cdot f(1) = 2 \cdot 1 = 2$, $f(3) = 3 \cdot f(2) = 3 \cdot 2 \cdot 1 = 6$, $f(4) = 4 \cdot f(3) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. This function is called the *factorial function*, and the function values are denoted usually by $x!$. Thus $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and so on. The factorial function is not one-to-one, for it contains two ordered pairs having 1 as 2nd coordinate.

The technique illustrated in Example G is called *definition by recursion* (or by *induction*). Without further elaboration we shall accept this technique as valid. For a discussion of inductive definition refer to Items 6 or 13 in the Reference List.

EXERCISES

1. Exhibit as a set of ordered pairs the function f defined on the domain I_5 by each of the following formulas:

(a) $f(x) = x$, (b) $g(x) = 6 - x$,

(c) $f(x) = 4$, (d) $g(x) = (-1)^x$,

(e) $f(x) = \sqrt{x}$, (f) $g(x) = 2^{-x}$,

(g) $f(1) = 1$ and $f(x) = x + f(x-1)$ for $1 < x \leqq 5$,

(h) $g(1) = 3$ and $g(x) = g(x-1) + (-1)^x$ for $1 < x \leqq 5$.

2. For each function in Exercise 1, sketch the graph as a set of five points on a cartesian coordinate system, and exhibit the range as a set of numbers.

3. Of the ten functions in Exercise 1, which of them are

(a) on I_5 onto I_5,

(b) on I_5 into I_5,

(c) recursively defined,

(d) one-to-one?

4. Let I represent the set of positive integers. A famous and useful function in the Theory of Numbers (called the *Euler phi function*) is defined as follows: for each positive integer x, $\phi(x)$ is the number of integers in the set $\{1, 2, \ldots, x\}$ having no divisor greater than 1 in common with x. Thus $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, since, for example, in the set $\{1, 2, 3\}$, 1 and 2 each have no divisor greater than 1 in common with 3.

(a) Find $\phi(5)$, $\phi(6)$, $\phi(7)$, $\phi(8)$, $\phi(9)$, $\phi(10)$.

(b) If p is a prime, show that $\phi(p) = p - 1$.

(c) Show that

(1) $\phi(12) = \phi(3)\cdot\phi(4)$ but $\phi(12) \neq \phi(2)\cdot\phi(6)$,

(2) $\phi(20) = \phi(4)\cdot\phi(5)$ but $\phi(20) \neq \phi(2)\cdot\phi(10)$,

(3) $\phi(24) = \phi(3)\cdot\phi(8)$ but $\phi(24) \neq \phi(4)\cdot\phi(6)$.

(d) Show that

(1) if a and b have a common divisor greater than 1, then $\phi(ab) \neq \phi(a)\cdot\phi(b)$,

(2) if a and b have no common divisor greater than 1, then $\phi(ab) = \phi(a)\cdot\phi(b)$.

5. Among all of the relations on I_2 into itself, how many are functions, and of these how many are one-to-one? Exhibit the matrix for each relation that is also a function.

6. In general, consider the matrix for a relation on I_n into itself.

(a) What matrix property guarantees that the relation be a function?

(b) What matrix property guarantees that the relation be a one-to-one function?

7. (a) Exhibit as relation matrices all functions on I_3 onto itself.

(b) Using some systematic procedure, count the functions on I_4 onto itself.

8. (a) Let S be the set of words {*zero*, *one*, *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*}. Exhibit the characteristic function on S for the property: the letter "*e*" fails to appear.

(b) Exhibit all characteristic functions on the set $\{p, q, r\}$. Referring to Example *E*, Chapter 1, relate (in some one-to-one way) these characteristic functions to the power set of $\{p, q, r\}$.

9. (a) If f is the factorial function, calculate the function values $f(6)$, $f(7)$, $f(8)$, $f(9)$, $f(10)$.

(b) Show that, in general,

(1) $(m\cdot n)! \neq (m!)(n!)$,

(2) $(m + n)! \neq m! + n!$.

(c) Calculate

(1) $10!/7!$, (2) $100!/96!$,

(3) $\dfrac{52!}{3!\,49!}$, (4) $7! - 6! + 5!$.

10. (a) Show that the product of the first n even positive integers is given by $2^n\cdot n!$.

(b) Derive a formula for the product of the first n odd positive integers.

The following problem material is provided for those who wish a broader introduction to the function concept than is necessary in the remainder of this study.

11. If f and g are functions defined on a common domain D, then

$$f + g = \{(x, y): x \in D, y = f(x) + g(x)\},$$
$$f - g = \{(x, y): x \in D, y = f(x) - g(x)\},$$
$$f \cdot g = \{(x, y): x \in D, y = f(x) \cdot g(x)\},$$
$$f/g = \{(x, y): x \in D, g(x) \neq 0, y = f(x)/g(x)\}.$$

Exhibit each of the four arithmetically defined functions above for the indicated pair of functions in Exercise 1:

(a) f as in (a) g as in (b),

(b) f as in (c) g as in (d),

(c) f as in (g) g as in (h).

12. Let f be a function defined on the domain D, and let A be a nonempty subset of D. The function $\{(x, f(x)): x \in A\}$ is called the *restriction of* f to A and is sometimes denoted by $f \mid A$. Note that $f \mid A \subset f$.
If A is the set $\{2, 4\}$, exhibit as a set of ordered pairs each function in Exercise 1 restricted to A.

13. Let $f: A \to B$ and $g: M \to N$ be two functions, and assume:

(i) f is on A onto B,

(ii) $B \cap M \neq \varnothing$,

(iii) $A^* = \{x: x \in A, f(x) \in M\}$.

The function $h: A^* \to N$ defined as

$$\{(x, h(x)): x \in A^*, h(x) = g(f(x))\}$$

is called the *composite of* g *and* f and is denoted by $g \circ f$. Although it is always the case that $f \cdot g = g \cdot f$ (see Problem 9), it is not always the case that $f \circ g = g \circ f$. For this reason, caution is required in the use of notation for the composite function.

(a) If we put $f^* = f \mid A^*$, observe that $g \circ f = g \circ f^*$.

(b) Exhibit the composition $g \circ f$, if it exists, for each indicated pair of functions in Exercise 1, and in each case determine A^* (as defined above):

(1) f as in (a) and g as in (b),

(2) f as in (c) and g as in (d),

(3) f as in (e) and g as in (f).

(c) Exhibit the composition $f \circ g$, if it exists, for each pair as given in (b) above.

(d) Exhibit the composition $g \circ g$ for g as defined in (b) of Exercise 1.

(e) Exhibit the composition $g \circ g$ for g as defined in (h) of Exercise 1.

14. Let A be a given set. A function $f: (A \times A) \to A$ is called a *binary operation on the set* A. Examples are the sum and product operations (of elementary arithmetic) on the positive integers. Binary operations may be exhibited (in part) by tables such as the addition and multiplication tables of elementary arithmetic. The domain of f consists of all ordered pairs (x, y) of elements in A and f is called a function of two variables. Although technically the function value corresponding to (x, y) should be written $f((x, y))$, it has become customary to write $f(x, y)$, since there is small danger of confusion.

(a) Let I be the set of positive integers. Define a binary operation f on I as follows: $f(x, y)$ is the largest prime factor of $x + y$. Then $f(1, 1) = 2$, $f(1, 2) = f(2, 1) = 3$, $f(2, 2) = 2$, $f(3, 1) = f(1, 3) = 2$, etc. Complete the table shown in Figure 4.5.

x \ y	1	2	3	4	5	6
1	2	3	2			
2	3	2				
3	2					
4						
5						
6						3

FIGURE 4.5

(b) If f is a binary operation on A and if for all x and y $f(x, y) = f(y, x)$, f is called a *commutative operation*. What tabular property corresponds to commutativity?

(c) Denote the subsets of $\{p, q, r\}$ as follows: $\varnothing = A_0$, $\{p\} = A_1$, $\{q\} = A_2$, $\{r\} = A_3$, $\{p, q\} = A_4$, $\{p, r\} = A_5$, $\{q, r\} = A_6$, $\{p, q, r\} = A_7$.

(1) Construct tables for the commutative binary operations $X \cup Y$ and $X \cap Y$.

(2) Construct the table for the non-commutative binary operation $X - Y$.

15. Let Z be a relation in the cartesian product $A \times B$. The set of ordered pairs

$$\{(x, y): (y, x) \in Z\}$$

is a relation in the cartesian product $B \times A$, and is called the *inverse of* Z. The inverse of a relation Z is obtained very simply by interchanging the 1st and 2nd coordinates of each ordered pair in Z. The inverse of Z is denoted by Z^{-1}. (Do not mistake this symbol for its use in arithmetic as an indication of the reciprocal of a number.)

(a) Describe a relation Z such that Z is a function but Z^{-1} is not a function.

(b) If Z is a one-to-one function, show that Z^{-1} is a function.

(c) Let Z be a relation on A into itself. What can be said about Z if $Z^{-1} = Z$?

Chapter Five

THE ART OF COUNTING

In the following definitions, the noun "power" is not to be confused with the adjective "power" as used in "power set." A synonym for "power" as used in this chapter is "cardinal number."

1 DEFINITION: *Two sets H and K are said to be of the* ***same power*** *iff there exists a one-to-one function on H onto K.*

2 DEFINITION: *A nonempty set H is said to be* ***finite*** *iff it is of the same power as some initial segment I_n. The* ***power*** *of I_n is n, and an arbitrary set H has* ***power*** *n iff H and I_n are of the same power. The empty set $\varnothing$ is finite and its power is zero.*

The power of a finite set is simply the number of elements in the set. Two finite sets having different numbers of elements are not of the same power; in particular, no finite set is of the same power as one of its proper subsets. This characteristic of finite sets provides a clue to a second method of defining infinite sets (the first relies on Definition 2: *a set is infinite iff it is not finite*).

3 DEFINITION: *A set M is said to be* ***infinite*** *iff there exists a proper subset T of M such that M and T are of the same power.*

In formal treatments of set theory it is proved that Definitions 2 and 3 are compatible. We shall assume this to be the case here.

A. The set $\{a, e, i, o, u\}$ is finite and its power is 5, since the function $\{(1, a), (2, e), (3, i), (4, o), (5, u)\}$ is one-to-one.

B. The set I of positive integers is infinite, for the function defined by $f(x) = x + 1$ with domain I and range $I - \{1\}$ is one-to-one and the range is a proper subset of the domain. There are many other one-to-one functions with domain I and range a proper subset of I: for example, the function defined by $f(x) = 2x$ in which the range is the set of even positive integers.

Let I_n be an initial segment with $n > 1$, let R be the set of all real numbers, and let $f: I_n \to R$ be a function. A useful notation for the sum and product of the n function values of f is

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + \cdots + f(n),$$

$$\prod_{x=1}^{n} f(x) = f(1) \cdot f(2) \cdot \cdots \cdot f(n).$$

Note that the terms, or factors, on the right need not be different numbers. In fact, if f is the constant function $f: I_n \to \{1\}$, then

$$\sum_{x=1}^{n} f(x) = n \quad \text{and} \quad \prod_{x=1}^{n} f(x) = 1.$$

The variable "x" appearing in this notation is sometimes called an *index letter*, and the range of x (that is, the set $\{1, 2, \ldots, n\}$) is sometimes called an *index set*. Other letters frequently used as index letters are i and j.

As a matter of convenience we agree that for any f,

$$\sum_{x=1}^{1} f(x) = \prod_{x=1}^{1} f(x) = f(1).$$

C. Let f be the identity function: it is defined by $f(x) = x$. Then

$$\sum_{x=1}^{5} f(x) = 1 + 2 + 3 + 4 + 5 = 15, \qquad \sum_{x=1}^{7} f(x) = 28,$$

$$\prod_{x=1}^{5} f(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5! = 120, \qquad \prod_{x=1}^{7} f(x) = 5040.$$

D. Consider the formula $f(i) = 2^i$. Then

$$\sum_{i=1}^{7} 2^i = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 = 254,$$

$$\prod_{i=1}^{7} 2^i = 2^1 \cdot 2^2 \cdot 2^3 \cdot 2^4 \cdot 2^5 \cdot 2^6 \cdot 2^7 = 2^{28}.$$

Problems related to counting will occupy much of our attention in the remainder of this study, but in problems of nonroutine interest simple enumeration is rarely sufficient. This part of our study belongs to a branch of mathematics known formally as *combinatorial analysis*, the tools and techniques of which depend largely upon the six theorems presented in this chapter. The proofs of three of these theorems require familiarity with mathematical induction and are deferred to the Appendix, where a brief discussion of induction is given.

4 THEOREM: *Let C be the union of two nonempty finite sets A and B. If $A \cap B = \varnothing$, then the power of C is the sum of the powers of A and B.*

> *Preliminary Remark:* The idea behind the proof is simply that we enumerate the elements first in one set followed by those in the other. In the enumeration we must insure that each element is counted, but none more than once.

Proof: Let

$f: A \to I_m$ be one-to-one and onto, and

$g: B \to I_n$ be one-to-one and onto,

so that the power of A is m and the power of B is n. We wish to prove that the power of C is $m + n$. Define a function $F: C \to I_{m+n}$ as follows:

$$F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ m + g(x) & \text{if } x \in B. \end{cases}$$

Then F is one-to-one and onto. ||

E. Let A be the set of vowels and let B be the set of consonants. Excluding the occasional use of y as a vowel, the power of A is 5 and the power of B is 21. Hence the power of $A \cup B$ is 26.

5 THEOREM: *Let n be a positive integer; let $A_0, A_1, \ldots, A_n$, be nonempty finite sets having powers $p_0, p_1, \ldots, p_n$, respectively; let $A_i \cap A_j = \varnothing$ for each $i \neq j$, and let C denote the union of all the sets. Then the power of C is $p_0 + p_1 + \cdots + p_n$.*

Proof: See the Appendix.

F. Consider the cartesian product $I_5 \times I_4$. Let A_0 be the subset consisting of those ordered pairs in which the 1st coordinate is 1, let A_1 be the subset consisting of those ordered pairs in which the 1st coordinate is 2, and so on. There are 5 such subsets, each of power 4, and none having points in common with another. The union is the cartesian product and its power is $4 + 4 + 4 + 4 + 4$.

Example F is generalized in the next theorem.

6 THEOREM: *Let A and B be nonempty finite sets. Then the power of the cartesian product $A \times B$ is the product of the powers of A and B.*

Preliminary Remark: To illustrate the idea behind the proof, we enumerate each set, $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. The cartesian product may be displayed in a rectangular array of m rows and n columns.

$$\begin{array}{llll} (a_1, b_1) & (a_1, b_2) & \ldots & (a_1, b_n) \\ (a_2, b_1) & (a_2, b_2) & \ldots & (a_2, b_n) \\ & \ldots & & \\ (a_m, b_1) & (a_m, b_2) & \ldots & (a_m, b_n) \end{array}$$

We then enumerate these ordered pairs by rows.

Proof: Let

$f: A \to I_m$ be one-to-one and onto, and

$g: B \to I_n$ be one-to-one and onto,

so that the power of A is m and the power of B is n. We wish to prove that the power of $A \times B$ is mn. Define a function $F: (A \times B) \to I_{mn}$ as follows:

$$F((a, b)) = n(f(a) - 1) + g(b)$$

for each $(a, b) \in (A \times B)$. The function F is one-to-one and onto since each integer in I_{mn} can be represented uniquely in the form $n(\alpha - 1) + \beta$, where $\alpha \in I_m$ and $\beta \in I_n$. ||

G. Let $A = \{p, q, r\}$ and let $B = \{a, b, c, d\}$. Consider the cartesian product as illustrated in Figure 5.1. Let $f = \{(p, 1), (q, 2), (r, 3)\}$ and let $g = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$. Notice that F enumerates the ordered pairs in $A \times B$ in rows. For example, $F((p, c)) = 4(1 - 1) + 3 = 3$, $F((q, a)) = 4(2 - 1) + 1 = 5$, and $F((r, d)) = 4(3 - 1) + 4 = 12$.

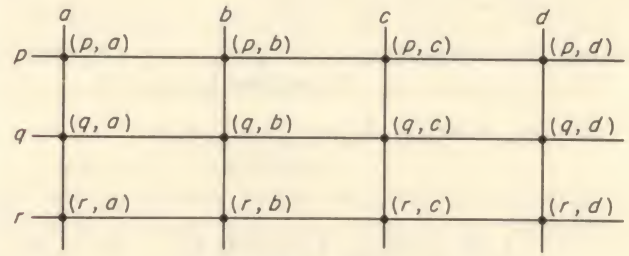

FIGURE 5.1

7 DEFINITION: *Let $n > 2$ and let $A_1, A_2, \ldots, A_n$, be nonempty sets. Let W be the set of all functions $f: I_n \to (A_1 \cup A_2 \cup \ldots \cup A_n)$. A subset T of W is called the* ***cartesian product*** *of the given sets A_x iff*

$$T = \{f: f \in W \text{ and for each } x \in I_n f(x) \in A_x\}.$$

Definition 7 yields an extension of the previously defined cartesian product of two sets. Points in the cartesian product T

just defined are sometimes called *n-tuples* and are denoted by $(a_1, a_2, \ldots, a_n)$ where each $a_x \in A_x$. This notation is simply an abbreviation for the function $\{(1, a_1), (2, a_2), \ldots, (n, a_n)\}$. The set T is sometimes denoted by $A_1 \times A_2 \times \cdots \times A_n$.

H. Let $A_1 = \{1, 2\}$, $A_2 = \{p, q\}$, and $A_3 = \{\theta, \pi\}$. Then

$$A_1 \times A_2 \times A_3 = \{(1, p, \theta), (1, q, \theta), (2, p, \theta), (2, q, \theta), (1, p, \pi), (1, q, \pi), (2, p, \pi), (2, q, \pi)\},$$

where each of the eight functions is expressed in the abbreviated form described in the preceding paragraph. Notice that the cartesian product is not necessarily commutative. In this example, any rearrangement of the factors produces a different cartesian product.

We may think of "constructing" the cartesian product T (in Definition 7) recursively as follows: Put $M_1 = A_1$, and for each i such that $1 \leqq i < n$ put $M_{i+1} = M_i \times A_{i+1}$. For each i, we agree to identify the point in M_i with the corresponding i-tuple as follows:

$i = 1, M_2 = A_1 \times A_2$: (a_1, a_2),

$i = 2, M_3 = M_2 \times A_3$: $((a_1, a_2), a_3)$ is written (a_1, a_2, a_3),

$i = 3, M_4 = M_3 \times A_4$: $((a_1, a_2, a_3), a_4)$ is written (a_1, a_2, a_3, a_4),

. . . . . .

$i = n - 1, M_n = M_{n-1} \times A_n$: $((a_1, a_2, \ldots, a_{n-1}), a_n)$ is written $(a_1, a_2, \ldots, a_{n-1}, a_n)$.

Then $T = M_n = M_{n-1} \times A_n$.

8 THEOREM: *Let n be a positive integer and let $A_0, A_1, \ldots, A_n$, be nonempty finite sets having powers $p_0, p_1, \ldots, p_n$, respectively. Then the power of $A_0 \times A_1 \times \cdots \times A_n$ is $p_0 \cdot p_1 \cdot \cdots \cdot p_n$.*

Proof: See the Appendix.

In Theorems 6 and 8, if each set in the cartesian products is A, and if the power of A is p, then the indicated products are simply powers of p. For example, the number of points in the cartesian product $A \times A$ is p^2 and the number of points in the cartesian product $A \times A \times A$ is p^3. The same result holds even though the sets may be different, so long as each set is of power p.

I. Of a group of 16 children, 7 were born in Alabama, 4 were born in Alaska, 3 were born in Arizona, and 2 were born in Arkansas. There are 168 ways in which we can select a set of four children so that each of the four states is represented. In the notation of Definition 7, each of these selections corresponds to a point in a cartesian product of sets having powers 7, 4, 3, and 2.

J. This sentence, excluding the nonletter symbols, may be considered a function $f: I_{100} \to A$, where A stands for the English alphabet. (Count the letters!) The number of such functions is 26^{100}. Such functions may be considered points in the cartesian product $A \times A \times \cdots \times A$, to 100 factors.

K. The number of functions $g: M \to M$, where M is nonempty and of power p, is p^p. This, of course, is the number of relations on M into itself which are also functions. A neat way to visualize this remark is as follows. Denote the elements in M by $a_1, a_2, \ldots, a_p$, and consider the set of ordered pairs

$$\{(a_1, \quad), (a_2, \quad), \ldots, (a_p, \quad)\}.$$

Each 2nd coordinate "blank" can be filled by any element in M (that is, in p ways). Thus any point in the cartesian product $M \times M \times \cdots \times M$ (p factors) describes a possible assignment of 2nd coordinates, and there are p^p such points.

L. The number of characteristic functions $k: M \to \{0, 1\}$, where M is nonempty and of power p, is 2^p. Consider again the set of ordered pairs $\{(a_1, \quad), (a_2, \quad), \ldots, (a_p, \quad)\}$. Here each 2nd coordinate "blank" can be filled by any element in the set $\{0, 1\}$.

As illustrated in the examples above, Theorem 8 solves the problem of enumerating all functions $f: I_n \to A$. We now investigate the enumeration of one-to-one functions on I_n into A.

If the power of A is p, then $n \leqq p$ for otherwise no one-to-one function can exist.

9 THEOREM: *Let A be a finite set of power $p > 1$. The number of one-to-one functions $f: I_2 \to A$ is $p(p-1)$.*

Proof: Let $x \in A$. The set $A - \{x\}$ has power $p - 1$, and the cartesian product $\{x\} \times (A - \{x\})$ has power $p - 1$ by Theorem 6. For each $x \in A$, let $A_x = \{x\} \times (A - \{x\})$. No two of the sets A_x have an element in common, hence by Theorem 5 the union of the p sets A_x has power $p(p-1)$. But the union of all the sets A_x is the set of all one-to-one functions on I_2 into A. ‖

M. Suppose there are five routes by which a person may travel between cities A and B. Then there are 20 round-trip routes between these cities in which the return differs from the route chosen in going.

10 THEOREM: *Let n be a positive integer. If A is a finite set of power $p > n$, then the number of one-to-one functions $f: I_{n+1} \to A$ is $p(p-1)(p-2)\cdots(p-n)$.*

Proof: See the Appendix.

In particular, if the power of A is p, then the number of one-to-one functions $f: I_p \to A$ is $p!$.

N. Let $A = \{w, x, y, z\}$. If t_n denotes the number of one-to-one functions of the form $f: I_{n+1} \to A$, then

$$t_1 = 4(4-1) = 12,$$

$$t_2 = 4(4-1)(4-2) = 24,$$

$$t_3 = 4(4-1)(4-2)(4-3) = 24.$$

Note that Theorem 10 excludes all $n > 3$.

EXERCISES

1. Let H and K be two sets and let $f: H \to K$ be one-to-one.
   (a) If f is onto, show that there is a one-to-one function $g: K \to H$ and that g is onto H.
   (b) If f is not onto, show that there is a proper subset K' of K such that there is a one-to-one function $g: K' \to H$ and that g is onto H.
2. There is a set of books on a shelf, each by a single author and no two by the same author. Among the authors so represented, four have names beginning with A, three with M, and three with R.
   (a) What is the power of the set of books?
   (b) What can be said about the power of the set of books if one of the conditions is altered to read "at most two are by any one author"?
3. Let A be a nonempty set and consider a function $f: I_8 \to A$ and three statements: (1) A contains 8 elements, (2) A contains at most 8 elements, (3) A contains at least 8 elements. Which of these three statements gives the most accurate information about A,
   (a) if f is one-to-one and onto?
   (b) if f is one-to-one?
   (c) if f is onto?
4. Let I be the set of positive integers. An infinite set M is called *countable* iff there is a one-to-one and onto function $f: I \to M$.
   (a) Let $A \cap B = \varnothing$ and suppose that A is finite and that B is infinite and countable. Show that $A \cup B$ is countable.
   (b) Let $A \cap B = \varnothing$ and suppose that A is infinite and countable and that B is infinite and countable. Show that $A \cup B$ is countable.
   (c) Show that the set of nonnegative integers is countable.
   (d) Show that the set of all integers is countable.
5. If A and B are finite, show that
   (power of $A \cup B$) = (power of A) + (power of B) − (power of $A \cap B$).
6. Prove, by mathematical induction, that the union of any finite number of finite sets is finite. (Assume no two of the sets have an element in common, and base your proof on Definition 2, not on Theorem 5.)
7. Prove Theorem 6 by direct appeal to Theorem 5.

8. In this exercise, let a "word" be a function $f: I_5 \to A$ (the English alphabet).

(a) How many words are there?

(b) How many words are there in which no letter is repeated?

(c) How many words are there that begin with a consonant and alternate vowels and consonants?

9. If x represents a three-place decimal, and if $0 < x < 1$, how many numbers are in the set to which x belongs?

10. How many numbers are there between 1000 and 10,000 in which no digit is repeated?

11. Let six points A, B, C, D, E, F lie on a circle. Beginning with any point, construct sequentially six segments connecting the points so that on each point there are exactly two segments. The six-sided figure so constructed is called a *hexagon.* How many different hexagons with these six vertices are there?

12. Let A be a finite set of power $n > 0$. Prove that $f: A \to A$ is onto iff it is one-to-one.

13. Let A be a finite set of power $n > 0$. How many functions are there on A into but not onto itself?

14. Let A be a finite set of power $n > 0$, and consider the cartesian product $A \times A$.

(a) How many functions are there in $A \times A$ with domain of power $n - 1$?

(b) How many one-to-one functions are there in $A \times A$ with domain of power $n - 1$?

15. As in Exercise 14, in $A \times A$ there are how many

(a) relations?

(b) reflexive relations?

(c) reflexive symmetric relations?

(d) symmetric relations?

(Hint. It may be helpful to consider relation matrices.)

Chapter Six

COMBINATIONS AND PERMUTATIONS

To every subset B of a set A there corresponds uniquely a characteristic function $k: A \to \{0, 1\}$ defined by

$$k(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

The two constant characteristic functions on A, $k: A \to \{0\}$ and $k: A \to \{1\}$ correspond, respectively, to the empty set and to A.

Let A be a set of power n, let B be a subset of A, and let k be the characteristic function corresponding to B. Then the sum of all function values $k(x)$ is the power of B, and we shall call this sum also the *power of the characteristic function* k. If the power of k is r, it is clear that $0 \leqq r \leqq n$.

1 DEFINITION: *A characteristic function on the finite set A is called a* ***combination on A.*** *If the characteristic function has power r, it is called a* ***combination of power r on A.***

A combination of power r on A (where A is of power n) is often called "a combination of r things out of n" or "a combination of n things taken r at a time."

2 DEFINITION: *If A is a set of power n, and if r is a positive integer such that $1 \leqq r < n$, then any one-to-one function $f\colon I_r \to A$ is called a* ***permutation of power r on A.*** *Any one-to-one function $f\colon I_n \to A$* is called a ***permutation on A.***

A permutation of power r on A is often called "a permutation of r things out of n" or "a permutation of n things taken r at a time."

The concepts introduced in Definitions 1 and 2 lie at the heart of combinatorial analysis, and certain enumeration formulas related to them are indispensable arithmetic tools in elementary probability theory.

The concept of "permutation" plays an important role in certain aspects of modern algebra. For this use, in particular, an alternate definition seems preferable: a *permutation on A is a one-to-one function on A onto itself.* How this definition is related to the one we have adopted will be considered in Exercises 9 and 10.

As a notational convenience, we shall use the symbols "$\{n\}_r$" and "$(n)_r$" for, respectively, the number of combinations of power r on A and the number of permutations of power r on A. Note that these symbols do not themselves represent combinations or permutations.

3 THEOREM:

$$(n)_r = \frac{n!}{(n-r)!}.$$

Proof: In Theorem 10 of Chapter 5 (page 42), replace p by n and $n+1$ by r. Then the number of permutations of power r on A is $n(n-1)\cdots(n-(r-1))$. But

$$\begin{aligned} n! &= n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1, \\ &= n(n-1)\cdots(n-r+1)\cdot(n-r)!. \end{aligned}$$

Hence, $(n)_r = n!/(n-r)!$. ‖

In particular, the number of permutations of power 1 on A is $(n)_1 = n!/(n-1)! = n$, and the number of permutations on A is $(n)_n = n!/(n-n)! = n!/0! = n!$. Referring to the definition of the factorial function (page 29), the reader may have wondered about the apparently artificial assignment of function values $0! = 1! = 1$. The motivation for this assignment lies in the simplification it affords in many of the formulas developed in this chapter. Although Definition 2 is not applicable when r is replaced by 0, we agree to accept the verdict of Theorem 3 and we assert that the number of permutations of power 0 on A is

$$(n)_0 = n!/(n-0)! = n!/n! = 1.$$

4 THEOREM:

$$\{n\}_r = \frac{n!}{r!(n-r)!}.$$

Proof: From the introductory remarks in this chapter, $\{n\}_r$ is the number of r-element subsets of the set A. For each of these subsets B, there are $r!$ permutations $f\colon I_r \to B$. The union of all of these permutations over all r-element subsets of A is the set of all permutations of power r on A. Hence by Theorem 5 of Chapter 5 (page 38)

$$\begin{aligned} (n)_r &= r! + r! + \cdots + r!, \qquad \text{to } \{n\}_r \text{ terms}, \\ &= r!\{n\}_r, \end{aligned}$$

from which the theorem follows. ‖

In particular,

$$\{n\}_0 = \{n\}_n = 1,$$

and

$$\{n\}_1 = \{n\}_{n-1} = n.$$

These special cases are transparent when the statements are translated into "subset" language. For example, $\{n\}_1$ is the number of singleton subsets of A.

A. In many parts of the country, local telephone numbers may be considered functions of the form $f\colon I_7 \to G$, where

$$G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

(1) There are 10^7 such functions.

(2) There are $(10)_7 = 604{,}800$ permutations of power 7 on G. That a telephone number is a permutation implies that no digit is dialed more than once.

(3) If neither 0 nor 1 appears among the first three function values, then the number of functions is $8^3 \cdot 10^4 = 5{,}120{,}000$.

(4) Among the functions enumerated in part (3), the number of those which are permutations of power 7 on G is

$$(8 \cdot 7 \cdot 6) \cdot (7 \cdot 6 \cdot 5 \cdot 4) = 282{,}240.$$

B. On the usual typewriter keyboard the 26 letters of the alphabet are arranged in three rows: 10 in the upper row, 9 in the middle row, and 7 in the lower row.

(1) The numbers of permutations of the letters in each row are, respectively, 10!, 9!, and 7!.

(2) The numbers of permutations of power 3 of the letters in each row are, respectively, $(10)_3 = 720$, $(9)_3 = 504$, and $(7)_3 = 210$.

(3) Let a permutation of power 3 on the set of letters in the alphabet be obtained by striking one key in each row. The three letters in any such permutation may be selected in $10 \cdot 9 \cdot 7$ ways, and for each selection the number of permutations of the three letters is $(3)_3 = 3! = 6$. Hence, the number of permutations described is $10 \cdot 9 \cdot 7 \cdot 6 = 3780$.

(4) Consider, now, permutations of power 3 in which two rows only are represented. In this case, two letters are selected from one row and the remaining letter may come from either of the other rows. The number of such permutations in which two letters are selected from the

upper row is $\{10\}_2 \cdot 16 \cdot 6 = 4320$,

middle row is $\{9\}_2 \cdot 17 \cdot 6 = 3672$,

lower row is $\{7\}_2 \cdot 19 \cdot 6 = 2394$.

The reader is urged to explain the fact that the sum of all calculated numbers in parts (2), (3), and (4) is 15,600, which is $(26)_3$.

5 THEOREM: *For* $0 \leqq r \leqq n$, $\{n\}_{n-r} = \{n\}_r$.

Proof:

$$\{n\}_{n-r} = n!/(n-r)!(n-(n-r))! = n!/(n-r)!n!. \quad \|$$

Note in Theorem 5 that for any r the sum of the subscripts is $n - r + r = n$.

6 THEOREM:

$$\sum_{i=0}^{n} \{n\}_i = 2^n.$$

Proof: Since $\{n\}_i$ is the number of characteristic functions of power i on a set A (of power n), $\sum_{i=0}^{n} \{n\}_i$ is the number of all characteristic functions on A. From Example L, Chapter 5, this number is 2^n. $\|$

In Chapter 5, we introduced the summation notation for the variable i on an initial segment. In Theorem 6, the notation is extended in the obvious way:

$$\sum_{i=0}^{n} \{n\}_i = \{n\}_0 + \sum_{i=1}^{n} \{n\}_i.$$

In any similar situation, the extended usage may be reduced to that in Chapter 5 by a simple change of index letter. For example, in Theorem 6, put $j = i + 1$. Then

$$\sum_{i=0}^{n} \{n\}_i = \sum_{j=1}^{n+1} \{n\}_{j-1}.$$

C. In each of six rooms in a certain house there is one wall switch controlling a ceiling light. Assume there are no other lights in these rooms.

(1) The number of ways in which exactly two of these rooms may be lighted is $\{6\}_2 = 6!/(2!4!) = 15$. Each way may be considered a characteristic function of power 2 on the set

of six rooms. The "on" position of the light switch corresponds to "1" and the "off" position to "0." It might be helpful to the reader to list all 15 of the possible lighted pairs of rooms. It is important to use some systematic procedure in producing the list, so that it will be clear that all pairs have been enumerated. One such scheme is: label the rooms a, b, c, d, e, f, then list all five pairs involving a, then all four pairs involving b but not a, etc.

(2) The number of ways in which exactly four of these rooms may be lighted is $\{6\}_4$. By Theorem 5, $\{6\}_4 = \{6\}_2 = 15$. Each selection of four lighted rooms yields a selection of two unlighted rooms. To each of these cases corresponds a case in Part (1) simply by reversing the six switches.

(3) By Theorem 6, the total number of ways these rooms may be lighted or not is $2^6 = 64$. This may be broken down as follows:

no rooms lighted . . .	$\{6\}_0 = 1$,
one room lighted . . .	$\{6\}_1 = 6$,
two rooms lighted . . .	$\{6\}_2 = 15$,
three rooms lighted . . .	$\{6\}_3 = 20$.

Use Theorem 5 for the other three cases, making a total of 64.

(4) The maximum number of combinations of r things out of 6 corresponds to the selection $r = 3$, as shown in part (3). If the possible values of r are listed in natural order $\{0, 1, 2, 3, 4, 5, 6\}$, the maximum value of $\{6\}_r$ occurs at the middle digit. Whether or not this is accidental is the question motivating our next theorem.

Let n be a positive integer and let f be the function defined on $\{0, 1, 2, \ldots, n\}$ by the formula $f(r) = \{n\}_r$. Now consider the ratio of two successive function values,

$$\frac{f(r)}{f(r+1)} = \frac{n!}{r!(n-r)!} \cdot \frac{(r+1)!(n-r-1)!}{n!} = \frac{r+1}{n-r}.$$

The graph of this ratio is shown in Figure 6.1 for $n = 6$ and in Figure 6.2 for $n = 7$. For increasing r between 0 and $n - 1$, the

numerator increases from 1 to n, while the denominator decreases from n to 1. Hence, as shown in the figures, the ratio increases from $1/n$ to n.

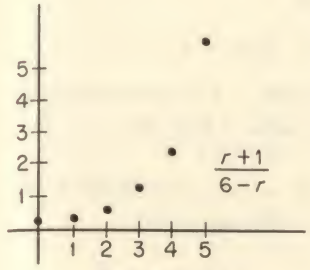

FIGURE 6.1

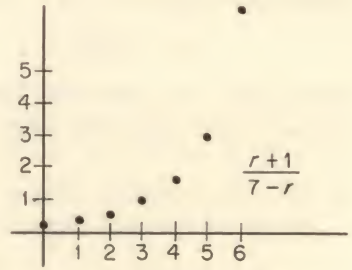

FIGURE 6.2

If the ratio is less than one, then $f(r) < f(r+1)$.
If the ratio is greater than one, then $f(r) > f(r+1)$.
If the ratio is one, then $f(r) = f(r+1)$.

These observations provide us with an essential clue in the following theorem, which continues the notation introduced in the preceding paragraphs.

7 THEOREM: *For a given positive integer n, let m be such that $n = 2m$ or $n = 2m + 1$. Then the maximum function value in f is $f(m)$. Furthermore, if n is even then $f(m) > f(r)$ for all $r \neq m$, and if n is odd then $f(m) = f(m+1)$ and $f(m) > f(r)$ for all r except m and $m + 1$.*

Proof: Let $n = 2m$. If $r = m - 1$ then

$$(r+1)/(2m-r) = m/(m+1) < 1,$$

hence $f(m-1) < f(m)$. If $r = m$ then

$$(r+1)/(2m-r) = (m+1)/m > 1,$$

hence $f(m) > f(m+1)$. Thus $f(m)$ is the maximum function value and it is greater than every other function value.

Let $n = 2m + 1$. If $r = m - 1$ then

$$(r+1)/(2m+1-r) = m/(m+2) < 1,$$

hence $f(m-1) < f(m)$. If $r = m$ then

$$(r+1)/(2m+1-r) = (m+1)/(m+1) = 1,$$

hence $f(m) = f(m+1)$. If $r = m+1$ then

$$(r+1)/(2m+1-r) = (m+2)/m > 1,$$

hence $f(m+1) > f(m+2)$. Thus $f(m) = f(m+1)$ is the maximum function value and every other function value is less than it. ‖

D. Let n be a positive integer and let $f(r) = \{n\}_r$. For each n the maximum function value in f occurs at the indicated r:

n:	1	2	3	4	5	6
r:	0, 1	1	1, 2	2	2, 3	3
n:	7	8	9	10	11	12
r:	3, 4	4	4, 5	5	5, 6	6

E. If it is known that n is such that $f(12)$ is the maximum function value, where $f(r) = \{n\}_r$, then if n is even it can only be 24, but if n is odd it can be either 23 or 25. This follows from Theorem 7. If n is even, put $m = 12$, from which $n = 2m = 24$. If n is odd, then either $m = 12$ or $m + 1 = 12$. In the first event, $n = 2m + 1 = 25$, and in the second event $n = 2m + 1 = 2(11) + 1 = 23$.

Let n be a positive integer and let K be a set of power $n+1$. Suppose

$$K = \{a_1, a_2, \ldots, a_n, a_{n+1}\} = A \cup \{a_{n+1}\}$$

where

$$A = \{a_1, a_2, \ldots, a_n\}.$$

Let r be any integer such that $0 \leqq r \leqq n$, and consider the combinations of power r on K. Let Q be the set of these combinations, and put $Q = Q_0 \cup Q_1$ where the combination k (of power r on K) is in Q_0 if $k(a_{n+1}) = 0$ and k is in Q_1 if $k(a_{n+1}) = 1$. The power of Q_0 is $\{n\}_r$; that is, the number of combinations of power r on A (since $k(a_{n+1}) = 0$). The power of Q_1 is $\{n\}_{r-1}$; that is, the number of combinations of power $r-1$ on A (since $k(a_{n+1}) = 1$).

The argument above suffices to prove the following theorem; nevertheless we shall include an arithmetic proof since it provides a good exercise in the manipulation of factorials.

8 THEOREM:

$$\{n+1\}_r = \{n\}_r + \{n\}_{r-1}.$$

Proof: By Theorem 4, the right-hand side of the equation can be written

$$\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!},$$

simplifying this expression leads to

$$\frac{(n-r+1)n! + r \cdot n!}{r!(n-r+1)!},$$

$$\frac{(n+1)n!}{r!(n+1-r)!},$$

$$\{n+1\}_r. \quad \|$$

We are now in a position to produce the familiar table (for n up to 6) shown in Figure 6.3. This table is called *Pascal's triangle*

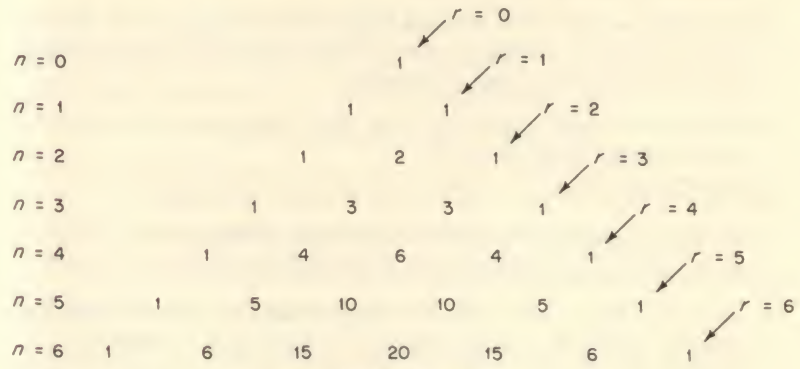

FIGURE 6.3

of coefficients in the expansion of the binomial $(x + y)^n$. Thus the numbers $\{n\}_r$ turn out to be the familiar binomial coefficients, which at the present time are denoted by a number of different symbols, perhaps more widely by $\binom{n}{r}$ than by any other. For several reasons we shall continue to use the symbol $\{n\}_r$.

9 THEOREM: *If n is any positive integer, then*

$$(x + y)^n = \sum_{i=0}^{n} \{n\}_{n-i} x^{n-i} y^i.$$

This is the *binomial theorem*; a proof of it is included in the Appendix.

EXERCISES

1. Out of a pile of twelve different books, nine are to be arranged on a shelf. In how many ways can this be done?
2. In how many ways can a committee of four be chosen from ten eligible individuals? In how many ways can the committee be chosen if two of the individuals desire to serve together on the committee or not at all?
3. A social committee of five members is to be chosen from a group of 12 girls and 10 boys. In how many ways can this be done if the committee is to have at most 3 boys?
4. Given the set $\{0, 1, 2, 3, 5\}$. How many positive integers have decimal representations consisting of three of these digits? How many of these are even? (Assume no digit is repeated.)
5. How many subsets containing more than one element are there of a set containing eleven elements?
6. If $n > 2$, let $\{P_1, P_2, \ldots, P_n\}$ be a set of points on a circle.
   (a) How many lines are there lying on pairs of these points?
   (b) How many triangles are there with vertices in the set?
7. A circular dinner table is set with eight places. Consider two seatings different only if the relative order of the individuals is different.
   (a) In how many ways can a set of eight people be seated at the table?

   (b) If the eight people consist of four men and four women, in how many ways can the people be seated so that men and women alternate?
   (c) If the eight people are made up of four couples, and if no couple is to sit side-by-side, in how many ways can the people be seated alternately men and women?
8. (a) Let A be a nonempty set of power n. If the subsets of A are classified according to size, what is the power of the subsets belonging to the most numerous class?
   (b) Referring to Theorem 7, if $f(r) = \{123\}_r$, for which r does f have a maximum function value?
   (c) For which n does $f(r) = \{n\}_r$ have a maximum function value at $r = 21$?
   (d) For fixed n, analyze the function $g(r) = (n)_r$ with respect to its maximum function value.
9. Let A be a nonempty set of power n. Let $f: I_n \to A$ be a permutation. Let $\mathcal{G}$ be the family of all permutations $g: I_n \to A$ (of course, this family includes f). Corresponding to each $g \in \mathcal{G}$, we define a function $g_f: A \to A$ as follows: if $x \in A$ and if $(i, x) \in f$, then $g_f(x) = g(i)$.
   (a) Show that g_f is on A onto A.
   (b) Show that f_f is the identity function on A.
   (c) Show that two different permutations $g: I_n \to A$ and $\phi: I_n \to A$ in $\mathcal{G}$ correspond to different functions g_f and ϕ_f.
10. Referring to Exercise 9, let $A = \{a, b, c, d, e\}$ and let

$$f = \{(1, a), (2, b), (3, c), (4, d), (5, e)\}.$$

   (a) If $g = \{(1, d), (2, c), (3, e), (4, b), (5, a)\}$, find g_f.
   (b) What permutation $g: I_5 \to A$ corresponds to

$$g_f = \{a, c), (b, b), (c, d), (d, e), (e, a)\}?$$

   (c) To what permutation does the identity function on A correspond?
11. Review and complete the proof of Theorem 8 using the argument begun in the paragraphs preceding the theorem.
12. Prove Theorem 9 by mathematical induction.
13. Using the binomial theorem, establish the result that the number of subsets of a set of power n is 2^n.

14. Let the set $\{(1, 0), (1/\sqrt{2}, 1/\sqrt{2}), (0, 1), (-1, 0), (0, -1)\}$ of points on the unit circle in the cartesian plane be called vertices. Let T be the set of all line segments having a pair of these vertices as end points.

(a) For each vertex, there are how many segments having it as one end point?

(b) How many segments are there in the set T?

(c) How many 4-element subsets of T are there?

(d) A 4-element subset of T is called a quadrilateral if each of its vertices is common to exactly two segments. How many quadrilaterals are there in T?

15. Generalize the preceding exercise by considering n points on the unit circle.

16. Let A be a nonempty finite set. How many functions f are there in $A \times A$ with domain a nonempty subset of A if

(a) the power of A is 3?

(b) the power of A is 4?

(c) the power of A is n?

Chapter Seven

INDISTINGUISHABLE ELEMENTS

Let A be a nonempty set of power n. In the last chapter we proved that the number of permutations on A is $(n)_n = n!$. A complication may arise if we ask, "Are these $n!$ permutations distinguishable from one another?"

A. Referring to Example A, Chapter 1, let the set of coins be denoted by $C = \{h, q, d_1, d_2, n\}$. The symbols d_1 and d_2 represent different coins which are, nevertheless, indistinguishable. For this reason, the permutations

$$\{(1, h), (2, q), (3, d_1), (4, d_2), (5, n)\}$$

and

$$\{(1, h), (2, q), (3, d_2), (4, d_1), (5, n)\},$$

for example, cannot be distinguished. In this way, the 5! permutations on C may be separated into $5!/2 = 60$ sets each containing two indistinguishable permutations.

The situation just illustrated may be generalized in the following way. Let $A = A_1 \cup A_2 \cup \cdots \cup A_k$, where

(1) $A_i \cap A_j = \varnothing$ if $i \neq j$,

(2) the power of A_i is t_i for each subscript and $t_i \geqq 1$,

(3) two elements in A are distinguishable if and only if they belong to different subsets.

Since the power of A is n, $t_1 + t_2 + \cdots + t_k = n$. There are $n!$ permutations on A, but not all of them are distinguishable.

1 THEOREM: *In the notation of the preceding paragraph, the number of distinguishable permutations on A is*

$$\frac{n!}{t_1! \cdot t_2! \cdot \cdots \cdot t_k!}.$$

Proof: Let x represent the number of distinguishable permutations on A. If, in fact, the elements of A_1 were all distinguishable, then the number of distinguishable permutations on A would be increased by the factor $t_1!$, since the number of permutations on A_1 is $t_1!$. Similarly, for each i if the elements of A_i were indeed distinguishable, then the number of distinguishable permutations on A would be increased by the factor $t_i!$ Thus we obtain the formula

$$x(t_1!t_2! \cdots t_k!) = n! \quad \|$$

The number of distinguishable permutations on A is affected only by the nonsingleton subsets A_j of A, for if $t_j = 1$, then $t_j! = 1$. Thus in Example A,

$$C = \{h\} \cup \{q\} \cup \{d_1, d_2\} \cup \{n\},$$

and the number of distinguishable permutations on C is $5!/2!$, omitting each $1!$ in the denominator.

If all elements in A are indistinguishable from one another, then the only distinguishable permutations or combinations are those of different powers. In the notation above, $A = A_1$ and the number of distinguishable permutations on A is $n!/n! = 1$.

B. To maintain modest order in his personal library, a certain teacher insists that books by the same author be shelved together. Of 11 books to be placed on a shelf, two are by author A, three by author B, and six by author C. There are $11!$ permutations on the set of 11 books, but since he is unconcerned about permutations of the subset of books by each author, there are $\frac{11!}{2!3!6!} = 4620$ distinguishable permutations. If the teacher is not concerned about alphabetic arrangement by author, then he considers six of these permutations orderly, and 4614 disorderly. If he desires alphabetic arrangement by author, then only one of the 4620 permutations is orderly. Note that the $11!$ permutations on the set of books may be subdivided into 4620 subsets within each of which the $2!3!6! = 8640$ permutations are indistinguishable. Of course, $(4620)(8640) = 39{,}916{,}800 = 11!$.

Throughout the remainder of this chapter we continue the notation suggested in the paragraph following Example A.

Two general problems now suggest themselves: (1) how many of the $(n)_r$ permutations of power r on A are distinguishable, and (2) how many of the $\{n\}_r$ combinations of power r on A are distinguishable?

For certain values assigned to r, these questions can be answered easily, but in full generality the problems are sufficiently complex that we shall not consider them here. A much fuller treatment may be found in several volumes included in the Reference List, particularly that by Riordan.

2 THEOREM: *The number of distinguishable permutations on A* (i) *of power* 1 *is k, and* (ii) *of power* $n - 1$ *is* $n!/(t_1!t_2! \ldots t_k!)$.

Proof: (i) This is simply the number of ways of selecting distinguishable elements in A, which is k since there are k nonempty subsets.

(ii) The number of distinguishable permutations of power $n - 1$ on A is the following sum of k terms:

$$\frac{(n-1)!}{(t_1 - 1)!t_2! \cdots t_k!} + \frac{(n-1)!}{t_1!(t_2 - 1)! \cdots t_k!} + \cdots + \frac{(n-1)!}{t_1!t_2! \cdots (t_k - 1)!}$$

The first term corresponds to the case in which the omitted element belongs to A_1, and in general the ith term corresponds to the case in which the omitted element belongs to A_i. Arithmetic simplification yields the perhaps unexpected result stated in the theorem. ‖

C. Consider the case in which $A = A_1 \cup A_2$ where $t_1 = 2$ and $t_2 = 3$. Let the indistinguishable elements in A_1 be denoted by x and those in A_2 by y. For simplicity, we shall use an ordered arrangement of letters to stand for a permutation: for example, $xxyyy$ stands for $\{(1, x), (2, x), (3, y), (4, y), (5, y)\}$. The number of distinguishable permutations on A is $5!/(2!3!) = 10$, and they are listed below in two sets, those in which the first element belongs to A_1 and those in which the first element belongs to A_2:

$$\{xxyyy, xyxyy, xyyxy, xyyyx\},$$
$$\{yxxyy, yxyxy, yxyyx, yyxxy, yyxyx, yyyxx\}.$$

The distinguishable permutations of power 4 on A may be found from the above list simply by dropping out the first letter in each case. The set of four corresponds to the term $4!/(1!3!)$ and the set of six corresponds to the term $4!/(2!2!)$.

As illustrated in Example C, we may draw the general conclusion that if A is separated into two subsets A_1 and A_2 then it necessarily follows that $t_2 = n - t_1$ and that the number of distinguishable permutations on A is simply the number of combinations of power t_1 on A. The reason is that each permutation is completely determined by assigning the positions in the ordered arrangement to be occupied by the elements in A_1, and this is the number of combinations of t_1 things out of n.

Recall that a combination on A is a characteristic function which defines a particular subset of A.

3 DEFINITION: *Two combinations on A are* ***indistinguishable*** *iff the subsets defined by the combinations contain the same number of elements in A_i for each i.*

4 THEOREM: *The number of distinguishable combinations of power r on A is $\{r + k - 1\}_r$, provided $r \leqq t_i$ for $1 \leqq i \leqq k$.*

Proof: Let r be less than or equal to the least of the powers of A_i. As a convenience we list the elements in A grouped from left to right in subsets $A_1, A_2, \ldots, A_k$, and between the subsets we place a separator "$*$" as follows:

$$\overbrace{a_1, a_1, \ldots, a_1}^{A_1} * \overbrace{a_2, a_2, \ldots, a_2}^{A_2} * \cdots * \overbrace{a_k, a_k, \ldots, a_k}^{A_k}.$$

A combination of power r assigns 1 to some r of the elements in A and 0 to the rest. Omitting all 0's, the combination then appears as a set of r 1's separated in some way by a set of $k - 1$ separators. For example, the symbol

$$1\ 1 * 1 * 1\ 1\ 1\ 1 * * 1 * \cdots$$

indicates that the combination selects 2 elements in A_1, 1 in A_2, 4 in A_3, none in A_4, 1 in A_5, and so on. Thus the distinguishable combinations of power r on A are the same as the distinguishable permutations on a set of power $r + k - 1$ in which r things are alike and $k - 1$ things are alike. Hence the number of such permutations is $\{r + k - 1\}_r$. ‖

The 0's which appear as function values in the combinations being enumerated provide no information essential to the problem. The power of A is not pertinent, so long as each subset A_i is of power at least r. This theorem applies, therefore, even though the power of A may not be known, nor even finite.

D. Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$, where each subset contains at least 5 elements. If $r = 5$, then the number of distinguishable combinations of power 5 on A is $\{5 + 4 - 1\}_5 = \{8\}_5 = 56$. A neat way to visualize any such combination is as a function in $I_5 \times I_4$. For example, let the elements in the subset of A selected by a given combination of power r be numbered, and let the point (i, j) indicate that the element numbered i is in the subset A_j. Since order of elements is irrelevant, we number them so that from left to right the 2nd coordinates never decrease. Figure 7.1 pictures the combination $1\ 1 * 1 * 1\ 1 *$; that is, 2 elements in A_1, 1 in A_2, 2 in A_3, and none in A_4. Now let $\alpha(i)$ denote the sum of the coordinates of the point

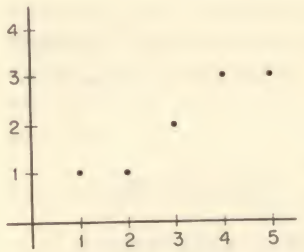

FIGURE 7.1

corresponding to the element numbered i. Then $\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 5, \alpha(4) = 7, \alpha(5) = 8$, and

$$\alpha(1) < \alpha(2) < \alpha(3) < \alpha(4) < \alpha(5).$$

For any such combination the minimum value of α is 2 and the maximum is $5 + 4$, and any combination is described by selecting from the set $\{2, 3, 4, 5, 6, 7, 8, 9\}$ the five α-values: under the convention that the α-values in increasing order correspond, respectively, to the elements numbered 1, 2, 3, 4, 5. Thus the number of such combinations is $\{8\}_5 = 56$.

E. A check is written so that it may be cashed with exactly five bills, none greater in amount than twenty dollars. Assuming that at least five bills in the amounts \$1, \$5, \$10, and \$20 are available, the number of ways such a check might be paid is $\{8\}_5 = 56$. This, of course, is simply a rephrasing of the previous example. Note that a check drawn in the amount of five dollars can be paid in only one way with five bills, but that a check drawn in the amount of fifty dollars can be paid in more than one way with five bills.

5 DEFINITION: *The function $f: I_m \to I_n$, in the cartesian product $I_m \times I_n$, is called* **isotone** *iff for each $a \in I_m$ and $b \in I_m$ if $a < b$ then $f(a) \leqq f(b)$.*

An isotone function is sometimes called *monotonic nondecreasing*. By reversing the inequality on the function values, we produce the definition of an *antitone* (or *monotonic nonincreasing*) function. The word *monotone* is reserved for functions which are either isotone or antitone. Note that a function may be both isotone and antitone: for example, a constant function such as $g: I_m \to \{1\}$.

These concepts usually occur in a much wider context: in fact, the essential requirement is that both the domain and the range be partially ordered.

Let $\mathscr{G}$ be the class of all functions $g: I_m \to I_n$. In Chapter 5 we

discovered that the number of such functions is n^m. We are now in a position to answer the question: how many of these are isotone?

F. Let $\mathscr{F}$ be the class of all isotone functions $f: I_m \to I_n$. (An example is shown in Figure 7.2.) Each such function is uniquely

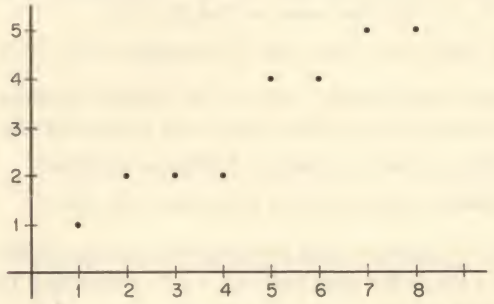

FIGURE 7.2

determined by the number of function values at each level $1, 2, \ldots, n$. The function graphed in Figure 7.2 can be described by the symbol "1 * 111 * * 11 * 11." By Theorem 4, the number of isotone functions $f: I_m \to I_n$ is $\{m + n - 1\}_m$.

EXERCISES

1. A freight train consists of 8 box cars, 5 tank cars, 5 gondolas, and 2 refrigerator cars. Considering cars of each type alike, in how many ways might the train be made up?
2. In the general notation used in this chapter, how many distinguishable permutations of power 2 are there on A, provided $2 \leqq t_i$ for $1 \leqq i \leqq k$?
3. In the general notation used in this chapter, let $r \leqq k$. If none of the k subsets of A can be represented by more than one element, how many distinguishable
   (a) permutations of power r on A are there?
   (b) combinations of power r on A are there?

4. Let A be a set of 10 chips, which are identical except for color. There are 4 white, 3 blue, and 3 red chips.

(a) The 10 chips are drawn one by one from a bowl, color being recorded at each draw. How many distinguishable drawings are there?

(b) If r chips are drawn at one time, what is the number of distinguishable combinations for $r = 0, 1, 2, 3, 4, 5$?

(c) For any j, show that the number of distinguishable combinations for $r = 10 - j$ is the same as that for $r = j$.

(d) Apply part (c) to extend part (b) through $r = 6, 7, 8, 9, 10$.

5. A company manufactures items of 20 different varieties. If these are offered for sale in sets of three items, how many such sets are there if

(a) the items are not necessarily of different varieties?

(b) no variety is represented by more than one item?

6. Referring to Example C, find the number of distinguishable combinations of power r for each r such that $0 \leqq r \leqq 5$. (Note that Theorem 4 does not apply for $r > 2$.)

7. Grades of A, B, C, D, and F are recorded for each course. A student taking three courses may achieve one of how many distinguishable grade records?

8. A five-card hand dealt from an ordinary deck of playing cards is called a *flush* provided all five cards are in the same suit.

(a) In how many ways can a flush be dealt?

(b) In how many ways can a five-card hand be dealt?

(c) Disregarding face value (that is, distinguishing cards only by suit), how many five-card hands can be dealt, and how many of these are flushes?

9. Referring to Example D, plot the functions corresponding to the combinations described by

(a) $1 * 1\ 1\ 1 * * 1$,

(b) $* 1\ 1\ 1\ 1\ 1 * *$,

(c) $1\ 1\ 1 * * * 1\ 1$.

10. Referring to Exercise 9, what set of α-values corresponds to the combination in each part?

11. Generalize the argument illustrated in Example D.

12. Let A be a set of power 3 in which 2 elements are indistinguishable, so that we may denote A by $\{a_1, a_2, b\}$. Let $\mathscr{F}$ be the family of all functions f on A into itself. How many distinguishable functions are there in $\mathscr{F}$?

13. How many monotone functions on I_m into I_n are there?

14. Examine the problem of enumerating the one-to-one isotone functions on I_m into I_n.

Chapter Eight

PARTITIONS OF INTEGERS

Intuitively, a partition of the positive integer n into m parts corresponds to a representation of n as the sum of m positive integers. Two such representations are considered identical provided they consist of the same integers with the same multiplicities. For example, the seven partitions of 5 into one, two, three, four, and five parts may be indicated, respectively, by

$$5$$

$$1 + 4 \quad \text{and} \quad 2 + 3$$

$$1 + 1 + 3 \quad \text{and} \quad 1 + 2 + 2$$

$$1 + 1 + 1 + 2$$

$$1 + 1 + 1 + 1 + 1.$$

It is intended that the representations $3 + 2$ and $2 + 3$, for example, not indicate different partitions. Condition (2) of Definition 1 is imposed specifically to avoid this undesired duplication.

1 DEFINITION: *Let n be a given positive integer. For any*

integer m, $1 \leqq m \leqq n$, let $f\colon I_m \to I_n$ be a function subject to the conditions

(1) $\sum_{i=1}^{m} f(i) = n$, *and*

(2) *f is isotone.*

Then f is called a **partition** *of the positive integer n into m* **parts.**

The number of these partitions is denoted by $P_m(n)$, and the total number of partitions of n (into whatever number of parts) is denoted by $P(n)$. Thus

$$P(n) = \sum_{m=1}^{n} P_m(n).$$

For $m = 1$ there is exactly one partition: $\{(1, n)\}$; and for $m = n$ there is exactly one partition: $\{(1, 1), (2, 1), \ldots, (n, 1)\}$. Hence, for any n, $P_1(n) = P_n(n) = 1$.

A. We may distinguish the seven partitions of the integer 5 by listing the function values. Note in each case that the sum of the function values is 5.

$P_1(5) = 1$: $f(1) = 5$.
$P_2(5) = 2$: $f(1) = 1$, $f(2) = 4$.
$f(1) = 2$, $f(2) = 3$.
$P_3(5) = 2$: $f(1) = f(2) = 1$, $f(3) = 3$.
$f(1) = 1$, $f(2) = f(3) = 2$.
$P_4(5) = 1$: $f(1) = f(2) = f(3) = 1$, $f(4) = 2$.
$P_5(5) = 1$: $f(1) = f(2) = f(3) = f(4) = f(5) = 1$.

We turn now to the problem of calculating $P_m(n)$ in general. Contrary to the situation with respect to permutations and combinations, there appears to be no simple formula for the calculation of $P_m(n)$, nor indeed of $P(n)$. Certain rather elaborate recursion formulas are known, and from them it is possible to construct tables to any desired size for $P(n)$ and $P_m(n)$. We shall investigate only the recursion formula for $P_m(n)$.

As a convenience in the next theorem, we agree to put

$$P_k(j) = 0 \text{ if } j < k.$$

2 THEOREM: *Let m and n be given positive integers such that $m \leqq n$. Then* $P_m(n) = \sum_{k=1}^{m} P_k(n-m)$.

Proof: Corresponding uniquely to each partition f of n into m parts, there is a partition g of $n-m$ into k parts (for some $k \leqq m$). The correspondence may be defined by the formula $g(i) = f(i) - 1$ for $i \in I_m$. For example, the partition $1 + 2 + 2$ of 5 into 3 parts corresponds to the partition

$$1 + 1(=(1-1) + (2-1) + (2-1))$$

of $5 - 3$ into 2 parts. Hence we find $P_m(n)$ by counting all partitions of $n - m$ into not more than m parts. ‖

Table 1 has been constructed by use of this theorem together

TABLE 1 For $P_m(n)$

n \ m	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	1	1	0	0	0	0
4	1	2	1	1	0	0	0
5	1	2	2	1	1	0	0
6	1	3	3	2	1	1	0
7	1	3	4	3	2	1	1

with the special cases mentioned earlier. By calculating the row sums in Table 1, we obtain the first seven values of $P(n)$, as listed in Table 2.

TABLE 2 For $P(n)$

n	1	2	3	4	5	6	7
$P(n)$	1	2	3	5	7	11	15

B. There are 15 experiment tables in a laboratory which must be used during one period by a class of 22 students. We are concerned only with the "pattern" of occupancy: that is, for any given assignment, the number of tables with zero occupants, with one occupant, with two occupants, and so on. Under these conditions, the number of possible assignments is the number of partitions of 22 into not more than 15 parts: $\sum_{m=1}^{15} P_m(22)$. We might consider the extreme case to be that in which all students are assigned to one table, corresponding to the partition $\{(1, 22)\}$. For some experiments, the optimum assignment might result from as few students as possible at each table. This situation corresponds to the partition

$$\{(1, 1), (2, 1), \ldots, (8, 1), (9, 2), (10, 2), \ldots, (15, 2)\}.$$

It is interesting to observe that the number of assignments found in Example B is, by Theorem 2, $P_{15}(37)$.

The ideas developed here and in earlier chapters can be used in analyzing the multinomial expansion, which is a generalization of the expansion of the binomial $(x_1 + x_2)^n$ (see Theorem 9, Chapter 6).

For abbreviation, put $X = x_1 + x_2 + \cdots + x_m$ and consider the expression $X^n = X_1 X_2 \cdots X_n$ where each $X_i = X$. The same m variables occur in each of the n factors X_i. The number of ways of selecting one variable from each of the n factors X_i is m^n, and corresponding to each selection we form the product of the n variables (one such product might be: $x_3 x_1 x_m \cdots x_2$). Then X^n is the sum of these products formed from the m^n possible selections. Thus the expansion of X^n involves a sum of m^n terms, but not all terms are distinct. For example, two selections might differ only in the first two factors, one being $x_3 x_1 \cdots$ and the other $x_1 x_3 \cdots$, from which by commutativity $x_3 x_1 \cdots = x_1 x_3 \cdots$.

By an argument similar to that used in Example D, Chapter 7, the number of distinguishable terms in the expansion is

$$\{n + m - 1\}_n.$$

Each term is of the form

$$T = x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m},$$

where $t_i \geqq 0$ and $\sum_{i=1}^{m} t_i = n$. Thus the nonzero exponents arranged in nondecreasing order describe a partition of n into at most m parts. Let this partition be represented by $\alpha_1 + \alpha_2 + \cdots + \alpha_k$. By Theorem 1, Chapter 7, the number of terms in the expansion of X^n of the form T is

$$\frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_k!},$$

since α_1 of the variables are alike, α_2 are alike, and so on. Since $0! = 1$, this number also may be calculated by the formula

$$\frac{n!}{t_1!t_2!\cdots t_k!}.$$

C. Consider the expansion of $(x_1 + x_2 + x_3)^4$. The number of terms with unit coefficient is $3^4 = 81$, and the number of distinguishable terms is $\{4 + 3 - 1\}_4 = 15$. The distinguishable terms may be constructed by concentrating on the exponents, which in each term must add to 4. For convenience, let $x_1 = x$, $x_2 = y$, and $x_3 = z$, and let the general term in the expansion of $(x + y + z)^4$ be $x^a y^b z^c$. The partitions of 4 into at most 3 parts are: (1) 4, (2) 1 + 3, (3) 2 + 2, (4) 1 + 1 + 2. Typical terms in the expansion related to these partitions are: (1) $x^4y^0z^0$, (2) $x^1y^3z^0$, (3) $x^2y^2z^0$, (4) $x^1y^1z^2$. In each case, every permutation of the exponents produces a distinguishable term. Hence, by Theorem 1, Chapter 7, there are (1) 3!/(1!2!), (2) 3!, (3) 3!/(2!1!), (4) 3!/(2!1!), distinguishable terms in each case. The exponents of these terms are tabulated in Table 3.

TABLE 3

(1)			(2)			(3)			(4)		
a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
4	0	0	1	3	0	2	2	0	1	1	2
0	4	0	1	0	3	2	0	2	1	2	1
0	0	4	3	1	0	0	2	2	2	1	1
			3	0	1						
			0	1	3						
			0	3	1						

Each distinguishable term related to a given partition has the same coefficient in the expansion: (1) 4!/4! = 1, (2) 4!/(1!3!) = 4, (3) 4!/(2!2!) = 6, (4) 4!/(1!1!2!) = 12. The expansion is, therefore,

$$\begin{aligned}(x + y + z)^4 = {} & x^4 + y^4 + z^4 + 4xy^3 + 4xz^3 + 4x^3y \\ & + 4x^3z + 4yz^3 + 4y^3z + 6x^2y^2 + 6x^2z^2 \\ & + 6y^2z^2 + 12xyz^2 + 12xy^2z + 12x^2yz.\end{aligned}$$

EXERCISES

Referring to the paragraph following Definition 7 of Chapter 5, express the partitions called for in these exercises as ordered m-tuples. For example, the partitions of 5 into 3 parts may be expressed as the ordered triples (1, 1, 3) and (1, 2, 2); and the ordered "2-tuple" (1, 4) represents the partition 1 + 4.

1. (a) List all partitions of 6.

(b) List all partitions of 7.

(c) List all partitions of 8.

2. Using the information found in Exercise 1, give some thought to the general problem of enumerating the partitions of n knowing those of $n - 1$. Estimate the number of partitions of 9 and 10, then check your estimates against the numbers given in the *Answers*.

3. List all partitions of the integer n into 5 parts for (a) $n = 9$, (b) $n = 10$, (c) $n = 11$, (d) $n = 12$.

4. List all partitions of the integer n into $n - 5$ parts for (a) $n = 9$, (b) $n = 10$, (c) $n = 11$, (d) $n = 12$.

5. A set M is said to be *partitioned* into k subsets $M_1, M_2, \ldots, M_k$ iff (1) $M_i \neq \varnothing$, (2) $M = M_1 \cup M_2 \cup \cdots \cup M_k$ and (3) $M_i \cap M_j = \varnothing$ for all $i \neq j$. If M contains n indistinguishable elements, in how many ways can M be partitioned into k nonempty subsets?

6. Referring to the notion of "relation matrix" in Chapter 3, how many distinguishable equivalence relations are there on a set M onto itself, if M contains

(a) 2 indistinguishable elements?

(b) 3 indistinguishable elements?

(c) n indistinguishable elements?

7. Discuss Exercise 6, if the elements in M are distinguishable from one another.

8. As in Example C, analyze the expansion of

$$(t + x + y + z)^3.$$

9. As in Example C, analyze the expansion of

$$(x + y + z)^5.$$

10. Just as the coefficients in the binomial expansion are called binomial coefficients, the coefficients in the multinomial expansion are called *multinomial coefficients.* Find the sum of the multinomial coefficients in the expansion of

$$(x_1 + x_2 + \cdots + x_m)^n.$$

11. In Definition 1, let condition (2) be eliminated. In effect, f is then an *ordered partition*, in which order of the parts is a distinguishing feature. How many ordered partitions $f: I_m \to I_n$ of n into m parts are there?

12. Referring to Exercise 11, how many ordered partitions of n (into whatever number of parts) are there?

Chapter Nine

THE OCCUPANCY PROBLEM

The general occupancy problem concerns the distribution of some finite number of objects among some finite number of containers: for example, the distribution of n passengers among the k cars of a subway train, or the distribution of n children among the k houses in a subdivision, or the distribution of n students among the k rooms in a dormitory. For uniformity of language, we shall use "elements" to refer to the objects or things distributed, and "cells" to refer to the containers among which the elements are distributed. The question is: in how many distinguishable ways can n elements be distributed among k cells?

In its general form the occupancy problem encompasses an entire class of more specialized problems, some of which occur when the elements and/or the cells are subjected to certain constraints. These constraints, aside from the numbers n and k, are of three kinds, related to the questions:

(1) Can the cells be distinguished from one another?

(2) Can the elements be distinguished from one another?

(3) Is there a limitation on multiple occupancy of a cell?

In this chapter we shall discuss only five major problems, each of which is related directly to our work in earlier chapters.

9.1 CONSTRAINTS: (1) cells distinguishable,
(2) elements distinguishable,
(3) any number of elements per cell.

A distribution of elements among the cells is completely specified when to each element a corresponding cell is assigned. We assume that the set of elements is indexed, $\{a_1, a_2, \ldots, a_n\}$, and we define a function on I_n by assigning to i that cell occupied by the element a_i. By condition (3) there are k possible function values for each i, hence the number of such functions (and, therefore, distributions) is k^n.

A. Any characteristic function on a finite set (for example,

$$\{(a, 1), (e, 1), (i, 0), (o, 0), (u, 0)\})$$

corresponds to a distribution of n elements among two cells, one labeled "0" and one labeled "1." The elements are distinguishable, the cells are distinguishable, and there is no limitation on multiple occupancy. The number of distributions is, of course, 2^n.

9.2 CONSTRAINTS: (1) cells distinguishable,
(2) elements distinguishable,
(3) at most one element per cell.

No distribution is possible unless $k \geqq n$. A distribution is completely specified when to each element a corresponding cell is assigned. Again, we assume that the set of elements is indexed, $\{a_1, a_2 \ldots, a_n\}$, and we define a function on I_n by assigning to i that cell occupied by the element a_i. By condition (3) this function must be one-to-one. Thus we are concerned with the number of permutations of n things (the n occupied cells) out of k, and this number is

$$(k)_n = k!/(k - n)!.$$

Of course, in the event that $k = n$, the number of distributions is $n!$.

B. Let the elements be the nonzero digits $\{1, 2, \ldots, 9\}$. Let p be a positive integer less than 1,000,000,000,000 constructed as follows: nine of the twelve possible places are filled with the nine digits, each being used once, and 0 is placed in each unoccupied position. Omitting initial zeros, each such number consists of either nine, ten, eleven, or twelve places. Altogether there are

$$(12)_9 = 12!/3! = 79{,}833{,}600 \text{ such numbers.}$$

9.3 CONSTRAINTS: (1) cells distinguishable,
(2) elements indistinguishable,
(3) at most one element per cell.

No distribution is possible unless $k \geqq n$. A distribution is completely specified when the occupied cells are known. Since there are n elements, conditions (2) and (3) imply that each distribution is a combination of power n on the set of k cells. Thus the number of distributions is

$$\{k\}_n = \frac{k!}{n!(k - n)!}.$$

In the event that $k = n$, there is exactly one distribution: each cell is occupied.

C. In *binary arithmetic*, the only nonzero "digit" is 1. In binary notation, a number is expressed as a sum of powers of 2; for example,

$$\begin{aligned} 11011 &= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 16 + 1 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1 \\ &= 27 \text{ (in decimal notation).} \end{aligned}$$

In each of the numbers constructed in Example B, let each nonzero digit be replaced by 1 and let 0 be retained in each of its occurrences. Altogether there are

$$\{12\}_9 = (12)_9/9! = 660 \text{ such numbers.}$$

9.4 CONSTRAINTS: (1) cells distinguishable,
(2) elements indistinguishable,
(3) any number of elements per cell.

A distribution is completely specified when to each cell a corresponding nonnegative integer is assigned to represent the number of elements in that cell. The sum of these numbers must be n. To determine the number of distributions we might first consider all partitions of n together with the distributions associated with each partition. Fortunately, this unsatisfactory scheme may be replaced by a simple observation. Cells may be denoted by the space beside or between separators "*," and each element may be indicated by a "1." A distribution of elements among the cells is indicated by placing 1's beside or between the separators. For example, four elements may be distributed among five cells by placing two elements in cell 3, one in cell 1, one in cell 4, leaving cells 2 and 5 empty. This distribution may be indicated by

$$1 * * 11 * 1 *$$

Cells: – 1 2 3 4 5

In general, k cells are indicated by $k - 1$ separators, and if there are n elements then there are $n + k - 1$ positions indicated. The assignment of 1's to any n of these $n + k - 1$ positions determines the distribution. Thus the number of such distributions is

$$\{n + k - 1\}_n.$$

D. An employer sets aside one hundred dollars to be given (in whole dollar amounts) as a Christmas bonus to five employees. In this situation one dollar is as good as any other, hence the dollars may be considered indistinguishable. On the other hand, the employees are clearly distinguishable. Although not all distributions would be in equal favor with all employees, there are $\{104\}_4 = 4{,}598{,}126$ distinguishable ways of distributing the bonus money.

9.5 CONSTRAINTS: (1) cells indistinguishable,
(2) elements indistinguishable,
(3) any number of elements per cell.

A distribution is completely specified by indicating the number of empty cells, the number of cells containing one element, the number of cells containing two elements, and so on. Since the cells are indistinguishable, in any distribution the nonempty cells may be indexed so that the function assigning to each i in some initial segment the number of elements in the cell c_i is isotonic. Thus each distribution is a partition of n into not more than k parts, hence the number of distributions is $\sum_{i=1}^{k} P_i(n)$.

E. On page 13 of a certain book, there are 14 lines which, for the purposes of this example, we assume to be indistinguishable. There are on this page 40 occurrences of the letter "*o*." The distribution of *o*'s among the 14 lines is, in fact: three lines with no *o*, three lines with one, two lines with two, three lines with four, two lines with six, and one line with nine. The number of distinguishable (but hypothetical) distributions of 40 *o*'s among 14 lines is $\sum_{i=1}^{14} P_i(40)$.

In addition to its wide distribution in mathematics, the occupancy problem has assumed an important place in modern atomic physics. According to presently accepted theories, the cases described above as 9.3 and 9.4 do, in fact, correspond closely to the known behavior of certain elementary particles. A discussion of this application may be found in William Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, 2nd ed. (New York: John Wiley & Sons, Inc., 1957) (see especially pages 20 and 38 ff.).

EXERCISES

1. To what theorem in the preceding chapters is each problem discussed in this chapter directly related?

Discuss the occupancy problem under each of the following constraints. In Exercises 4, 5, and 6 do not expect to find a general formula.

2. (1) Cells indistinguishable,
(2) elements indistinguishable,
(3) at most one element per cell.

3. (1) Cells indistinguishable,
(2) elements distinguishable,
(3) at most one element per cell.

4. (1) Cells indistinguishable,
(2) elements distinguishable,
(3) any number of elements per cell.

5. (1) Cells distinguishable,
(2) elements distinguishable,
(3) at most two elements per cell.

6. (1) Cells distinguishable,
(2) elements indistinguishable,
(3) at most two elements per cell.

Appendix

We assume on the part of the reader a reasonable familiarity with mathematical induction, but in order to renew acquaintance (if necessary) we state one form of the induction axiom and prove two theorems by way of illustration.†

1 AXIOM: *Let I be the set of all positive integers, and suppose $M \subset I$. If both*

(1) $1 \in M$, *and*
(2) $k \in M$ *implies* $k + 1 \in M$,

then $M = I$.

2 THEOREM: *Let $C(n)$ denote the equation*

$$\sum_{i=1}^{n} i = n(n + 1)/2.$$

Then for each positive integer n, $C(n)$ is true.

Proof: Let $M = \{n : n \in I \text{ and } C(n) \text{ is true}\}$.

(1) $1 \in M$, since $\sum_{i=1}^{1} i = 1 = 1(1 + 1)/2$.

† See item 13 in the Reference List.

(2) By definition of the sigma notation

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1),$$

and on the assumption that $k \in M$ we may replace $\sum_{i=1}^{k} i$ by $k(k+1)/2$. Arithmetic simplification yields

$$\sum_{i=1}^{k+1} i = (k+1)(k+2)/2,$$

which implies that $k+1 \in M$. By the axiom we conclude that $M = I$. ‖

3 THEOREM: *Let $C(n)$ denote the equation*

$$\sum_{i=1}^{n} 2^i = 2^{n+1} - 2.$$

Then for each positive integer n, $C(n)$ is true.

Proof: Let $M = \{n: n \in I \text{ and } C(n) \text{ is true}\}$.

(1) $1 \in M$, since $\sum_{i=1}^{1} 2^i = 2 = 2^2 - 2$.

(2) By definition of the sigma notation

$$\sum_{i=1}^{k+1} 2^i = \sum_{i=1}^{k} 2^i + 2^{k+1},$$

and on the assumption that $k \in M$ we may replace $\sum_{i=1}^{k} 2^i$ by $2^{k+1} - 2$. Arithmetic simplification yields

$$\sum_{i=1}^{k+1} 2^i = 2^{k+2} - 2,$$

which implies that $k+1 \in M$. By the axiom we conclude that $M = I$. ‖

A close connection can be observed between these theorems and several parts of Examples C and D of Chapter 5.

The remainder of this appendix is devoted to four theorems,

proofs of which were omitted in the main body of the text. The theorems are restated here for convenience in slightly altered form.

4 THEOREM (Chapter 5, No. 5): *Let $C(n)$ denote the equation*

$$\text{power of } (A_0 \cup A_1 \cup \cdots \cup A_n) = p_0 + p_1 + \cdots + p_n,$$

where $A_0, A_1, \ldots, A_n$ are finite nonempty sets having powers $p_0, p_1, \ldots, p_n$, respectively, and $A_i \cap A_j = \varnothing$ for each $i \neq j$. Then for each positive integer n, $C(n)$ is true.

Proof: Let $M = \{n: n \in I \text{ and } C(n) \text{ is true}\}$.

(1) $1 \in M$, by Theorem 4 of Chapter 5.

(2) Let $A_0, A_1, \ldots, A_{k+1}$, be sets satisfying the conditions stated in the theorem, and let $B = A_0 \cup A_1 \cup \cdots \cup A_{k+1}$. Then

$$B = (A_0 \cup A_1 \cup \cdots \cup A_k) \cup A_{k+1},$$

and on the assumption that $k \in M$ we may replace the power of $A_0 \cup A_1 \cup \cdots \cup A_k$ by $p_0 + p_1 + \cdots + p_k$. By another application of Theorem 4,

$$\text{power of } B = (p_0 + p_1 + \cdots + p_k) + p_{k+1},$$

which implies that $k+1 \in M$.

By the axiom we conclude that $M = I$. ‖

5 THEOREM (Chapter 5, No. 8): *Let $C(n)$ denote the equation*

$$\text{power of } (A_0 \times A_1 \times \cdots \times A_n) = p_0 p_1 \cdots p_n,$$

where $A_0, A_1, \ldots, A_n$ are finite nonempty sets having powers $p_0, p_1, \ldots, p_n$, respectively. Then for each positive integer n, $C(n)$ is true.

Proof: Let $M = \{n: n \in I \text{ and } C(n) \text{ is true}\}$.

(1) $1 \in M$, by Theorem 6 of Chapter 5.

(2) Let $A_0, A_1, \ldots, A_{k+1}$ be sets satisfying the conditions stated in the theorem, and let $B = A_0 \times A_1 \times \cdots \times A_{k+1}$. We may identify B with the product $(A_0 \times A_1 \times \cdots \times A_k) \times A_{k+1}$, and on the assumption that $k \in M$ we may replace the power of

$A_0 \times A_1 \times \cdots \times A_k$ by $p_0 p_1 \cdots p_k$. By another application of Theorem 6,

$$\text{power of } B = (p_0 p_1 \cdots p_k) p_{k+1},$$

which implies that $k + 1 \in M$.

By the axiom we conclude that $M = I$. ‖

6 THEOREM (Chapter 5, No. 10): *Let $C(n)$ denote the equation*

$$\text{power of } F_n(A) = p(p-1)(p-2)\cdots(p-n),$$

where A is a finite set of power $p > n$ and $F_n(A)$ is the set of all one-to-one functions f: $I_{n+1} \to A$. Then for each positive integer n, $C(n)$ is true.

Proof: Let $M = \{n: n \in I \text{ and } C(n) \text{ is true}\}$.

(1) $1 \in M$, since $F_1(A)$ is the set of all one-to-one functions $f: I_2 \to A$, and Theorem 9 of Chapter 5 applies.

(2) Assume that $p > k + 1$ and consider

$$F_{k+1}(A) = \{f: f \text{ is one-to-one and } f \text{ is on } I_{k+2} \text{ into } A\}.$$

Each function f in $F_{k+1}(A)$ is a set of ordered pairs

$$f = \{(1, a_1), (2, a_2), \ldots, (k, a_k), (k+1, a_{k+1}), (k+2, a_{k+2})\},$$

where the a_i are different elements in A. Thus

$$f = \{(1, a_1), (2, a_2), \ldots, (k, a_k), (k+1, a_{k+1})\} \cup \{(k+2, a_{k+2})\}.$$

For any given element $a_{k+2} \in A$, the first set on the right-hand side is a one-to-one function on I_{k+1} into $(A - \{a_{k+2}\})$. On the assumption that $k \in M$, the number of such functions is

$$(p-1)((p-1)-1)((p-1)-2)\cdots((p-1)-k)$$

$$= (p-1)(p-2)(p-3)\cdots(p-(k+1)).$$

Thus $F_{k+1}(A)$ can be represented as the union of p subsets, one for each of the p possible elements a_{k+2}, no two of the subsets having a function in common, and each subset being of power

$(p-1)(p-2)(p-3)\ldots(p-(k+1))$. By Theorem 5 of Chapter 5,

$$\text{power of } F_{k+1}(A) = p(p-1)(p-2)\cdots(p-(k+1)),$$

which implies that $k + 1 \in M$.

By the axiom we conclude that $M = I$. ‖

7 THEOREM (Chapter 6, No. 9): *Let $C(n)$ denote the equation*

$$(x+y)^n = \sum_{i=0}^{n} \{n\}_{n-i} x^{n-i} y^i.$$

Then for each positive integer n, $C(n)$ is true.

Proof: Let $M = \{n: n \in I \text{ and } C(n) \text{ is true}\}$.

(1) $1 \in M$, since $\sum_{i=0}^{1} \{1\}_{1-i} x^{1-i} y^i = x + y$.

(2) From elementary algebra, $(x+y)^{k+1} = (x+y)(x+y)^k$, and on the assumption that $k \in M$ we may replace $(x+y)^k$ by $\sum_{i=0}^{k} \{k\}_{k-i} x^{k-i} y^i$. Hence

$$(x+y)^{k+1} = (x+y)\left(\sum_{i=0}^{k} \{k\}_{k-i} x^{k-i} y^i\right)$$

$$= \sum_{i=0}^{k} \{k\}_{k-i} x^{k+1-i} y^i + \sum_{i=0}^{k} \{k\}_{k-i} x^{k-i} y^{i+1}.$$

Arithmetic simplification, using Theorem 8 of Chapter 6, leads to

$$(x+y)^{k+1} = \sum_{i=0}^{k+1} \{k+1\}_{k+1-i} x^{k+1-i} y^i,$$

which implies that $k + 1 \in M$.

By the axiom we conclude that $M = I$. ‖

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REFERENCE KEY

References are classified by chapter. The symbol "(x; y)" means "Chapter y of Item x" in the Reference List, except where indicated otherwise.

I (1; 2) (3; 1) (6; 4) (8; Appendix 1) (10; 1) (11; 1) (12; 1)
II (1; 2) (3; 1) (5; 2) (6; 4) (8; Appendix 1) (11; 1) (12; 1)
III (3; 1) (6; 4) (6; 5) (11; 3) (12; 1)
IV (6; 5) (6; 11) (11; 4) (12; 1) (13; 3)
V (1; 3) (2; 2) (5; 3) (6; 10) (8; Appendix 2)
VI (2; 2) (3; 3) (4; page 41 ff.) (5; 3) (7; 1) (8; 2) (9; 1) (10; 1)
VII (2; 2) (8; 2) (9; 1)
VIII (4; page 51 ff.) (7; 1) (9; 6)
IX (2; 2) (7; 2) (9; 5)
Appendix (13)

Answers

(Answers or hints are given for most exercises except those requiring figures.)

CHAPTER ONE

1. (a) $\varnothing$, $\{x\}$, $\{y\}$, $\{x, y\}$. (b) $\{x\}$, $\{y\}$. (c) $\varnothing$, $\{x\}$, $\{y\}$. (d) Incorrect parts are (2), (3), (6).

2. (a) $\{p, s\}$, $\{q, s\}$, $\{r, s\}$. (b) $\{p, q\}$, $\{p, r\}$ $\{q, r\}$. (c) Respectively, 1, 4, 6, 4, 1. (d) Incorrect parts are (2), (3), (4), (6), (9).

3. $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.

4. (a) $\{1, -1\}$. (b) $\{0\}$. (c) $\varnothing$.

5. (a) 0. (b) 1. (c) 2. (d) 3.

6. (a) $\{3, 7, 11\}$. (b) $\{1, 3, 7, 11, 21, 33, 49, 77, 147, 231, 539, 1617\}$.

7. (a) Hint. There are 16 subsets of $A = \{3, 5, 7, 11\}$. (b) To each non-empty subset of A corresponds the factor given by the product of its elements.

8. (a) 7. (b) $n - k$.

CHAPTER TWO

1. (a) $x = 2$. (b) $\{3, 5, 11, 17, 29, 41\}$. (c) No. (d) $S - \{2\}$ is the set of odd primes. (e) Respectively, 1, 1, 12, 35.

2. (a) $\{m + 1, m + 2, \ldots, n\} = \{x : x \in I \text{ and } m < x \leqq n\}$. (b) $\varnothing$. (c) $\varnothing$.

3. (a) $\{2, 3, 5, 7\}$. (b) $\{0, 1, 4, 6, 8, 9\}$. (c) $M - S$. (d) $\varnothing$. (e) Incorrect parts are (3), (6), (7).

4. The set is $\{4, 5, \ldots, 10\}$.

5. (a) $\{0\}$. (c) $0 \in (M - I_9)$ but $0 \notin (I_9 - M)$. (d) The set is $\{10\}$. (e) $0 \in (I_{12} \cup M)$ but $0 \notin I_{12}$.

7. If $x \in (M - N)$ then $x \in M$ and $x \notin N$, hence $x \in M$ and $x \in (S - N)$. This shows that $(M - N) \subset (M \cap (S - N))$. Now show that $(M \cap (S - N)) \subset (M - N)$.

8. (a) $\{0, 10\}$. (b) $\{4, 5, 6, 7\}$.

9. 52.

10. 15 per cent.

CHAPTER THREE

2. The equation $y = 0$ has, respectively, 2, 1, and 0 real roots.

3. (a) Respectively, 1, 9, 25. (b) The first eight values of $N(k)$ are 1, 9, 25, 45, 69, 109, 145, 193.

4. (a) 15. (b) 9. (c) Domain = $\{1, 2\}$, range = $\{2, 3, 4, 5\}$.

5. (b) Respectively, yes, no, yes.

7.

10	01	10	01	11	11	10	01	11	
10	01	01	10	01	10	11	11	11	

8.

100	110	101	100	111	101	110	111
010	110	010	011	110	011	111	111
001	001	101	011	101	111	011	111

CHAPTER FOUR

1. (a) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.
(b) $\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$.
(c) $\{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$.
(d) $\{(1, -1), (2, 1), (3, -1), (4, 1), (5, -1)\}$.
(e) $\{(1, 1), (2, \sqrt{2}), (3, \sqrt{3}), (4, 2), (5, \sqrt{5})\}$.
(f) $\{(1, 1/2), (2, 1/4), (3, 1/8), (4, 1/16), (5, 1/32)\}$.
(g) $\{(1, 1), (2, 3), (3, 6), (4, 10), (5, 15)\}$.
(h) $\{(1, 3), (2, 4), (3, 3), (4, 4), (5, 3)\}$.

2. The ranges are, respectively, $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 5\}$, $\{4\}$, $\{1\ -1\}$, $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}\}$, $\{1/2, 1/4, 1/8, 1/16, 1/32\}$, $\{1, 3, 6, 10, 15\}$, $\{3, 4\}$.

3. (a) ((a), (b)). (b) ((a), (b), (c), (h)). (c) ((g), (h)). (d) ((a), (b), (e), (f), (g)).

4. (a) Respectively, 4, 2, 6, 4, 6, 4. (b) Only p has a divisor greater than 1 in common with p. (c) Respectively, $\phi(12) = 4$, $\phi(20) = 8$, $\phi(24) = 8$.

5. Respectively, 4, 2.

10	01	10	01
10	01	01	10

6. (a) The sum of the entries in each row is 1. (b) The sum of the entries in each row and each column is 1.

7. (a)

100	100	010	010	001	001
010	001	100	001	100	010
001	010	001	100	010	100

(b) 24.

8. (a) {(zero, 0), (one, 0), (two, 1), (three, 0), (four, 1), (five, 0), (six, 1), (seven, 0), (eight, 0), (nine, 0)}.

(b) Let the numbers in each column be the function values for one characteristic function. The eight characteristic functions may be tabulated as follows:

p:	0	1	0	0	1	1	0	1
q:	0	0	1	0	1	0	1	1
r:	0	0	0	1	0	1	1	1

9. (a) Respectively, 720, 5040, 40,320, 362,880, 3,628,800.

(b) For example, $6! \neq 3!2!$ and $5! \neq 3! + 2!$.

(c) Respectively, 720, 94,109,400, 22,100, 4440.

10. (a) $2^n(1 \cdot 2 \cdot 3 \cdots \cdot n) = 2 \cdot 4 \cdot 6 \cdots \cdot (2n)$. (b) $(2n)!/2^n n!$.

11. Function values are tabulated, respectively,

	(a)				(b)				(c)			
1	6	−4	5	1/5	3	5	−4	−4	4	−2	3	1/3
2	6	−2	8	1/2	5	3	4	4	7	−1	12	3/4
3	6	0	6	1	3	5	−4	−4	9	3	18	2
4	6	2	8	2	5	3	4	4	14	6	40	5/2
5	6	4	5	5	3	5	−4	−4	18	12	45	5

12. (a) $\{(2, 2), (4, 4)\}$.
(b) $\{(2, 4), (4, 2)\}$.
(c) $\{(2, 4), (4, 4)\}$.
(d) $\{(2, 1), (4, 1)\}$.
(e) $\{(2, \sqrt{2}), (4, 2)\}$.
(f) $\{(2, 1/4), (4, 1/16)\}$.
(g) $\{(2, 3), (4, 10)\}$.
(h) $\{(2, 4), (4, 4)\}$.

13. (b) (1) $A^* = \{1, 2, 3, 4, 5\}$, $g \circ f = g$;
(2) $A^* = \{1, 2, 3, 4, 5\}$, $g \circ f = \{(x, 1): x \in A^*\}$;
(3) $A^* = \{1, 4\}$, $g \circ f = \{(1, 1/2), (4, 1/4)\}$.
(c) (1) $f \circ g = g$;
(2) $f \circ g = \{(2, 4), (4, 4)\}$;
(3) $f \circ g = \varnothing$.
(d) $g \circ g$ is the identity function.
(e) $g \circ g = g$.

14. (a)

2	3	2	5	3	7
3	2	5	3	7	2
2	5	3	7	2	3
5	3	7	2	3	5
3	7	2	3	5	11
7	2	3	5	11	3

(b) The table is symmetric about the main diagonal, as in the table of part (a).

(c) The three tables, respectively, indicate only the subscripts.

0 1 2 3 4 5 6 7	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0
1 1 4 5 4 5 7 7	0 1 0 0 1 1 0 1	1 0 1 1 0 0 1 0
2 4 2 6 4 7 6 7	0 0 2 0 2 0 2 2	2 2 0 2 0 2 0 0
3 5 6 3 7 5 6 7	0 0 0 3 0 3 3 3	3 3 3 0 3 0 0 0
4 4 4 7 4 7 7 7	0 1 2 0 4 1 2 4	4 2 1 4 0 2 1 0
5 5 7 5 7 5 7 7	0 1 0 3 1 5 3 5	5 3 5 1 3 0 1 0
6 7 6 6 7 7 6 7	0 0 2 3 2 3 6 6	6 6 3 2 3 2 0 0
7 7 7 7 7 7 7 7	0 1 2 3 4 5 6 7	7 6 5 4 3 2 1 0

15. (a) For example, any constant function with more than one element in the domain.

(b) If Z^{-1} is not a function then for some $x_1 \neq x_2$, $(y, x_1) \in Z^{-1}$ and $(y, x_2) \in Z^{-1}$. Hence $(x_1, y) \in Z$ and $(x_2, y) \in Z$ and Z is not one-to-one.

(c) Z is onto and is symmetric.

CHAPTER FIVE

1. (a) $f(H) = K$. The function $\{(x, y): (y, x) \in f\} = f^{-1}$ is one-to-one and onto.

(b) $f(H) = K'$ is a proper subset of K. The function $\{(x, y): (y, x) \in f\}$ is one-to-one and onto.

2. (a) 10. (b) At least 10 but not more than 20.

3. (a) (1). (b) (3). (c) (2).

4. (a) Let $g\colon I_n \to A$ and $f\colon I \to B$ each be one-to-one and onto. The function $F: I \to (A \cup B)$ defined by

$$F(x) = \begin{cases} g(x) & \text{if } x \leqq n, \\ f(x-n) & \text{if } x > n, \end{cases}$$

is one-to-one and onto.

(b) Let $f\colon I \to A$ and $g\colon I \to B$ each be one-to-one and onto. The function $F: I \to (A \cup \mathrm{B})$ defined by

$$F(x) = \begin{cases} f((x+1)/2) & \text{if } x \text{ is odd}, \\ g(x/2) & \text{if } x \text{ is even}, \end{cases}$$

is one-to-one and onto.

(c) $\{0, 1, 2, \ldots\} = \{0\} \cup \{1, 2, \ldots\}$ and apply part (a).

(d) $\{\ldots, -2, -1, 0, 1, 2, \ldots\} = \{0, 1, 2, \ldots\} \cup \{-1, -2, \ldots\}$ and apply part (b).

5. *Hint:* Assume the indicated sets are nonempty. Write $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$ and apply Theorem 4. Then $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$ and apply Theorem 5.

6. *Hint:* Review the Appendix if necessary.

7. *Hint:* For each $x \in A$ let $A_x = \{(x, y)\colon y \in B\}$.

8. (a) 26^5. (b) $26!/21!$. (c) $21^3 \cdot 5^2$.

9. 999.

10. 4536.

11. 60.

13. $n^n - n!$.

14. (a) n^n. (b) $n \cdot n!$.

15. Since each square in the n by n relation matrix contains either "0" or "1," each answer is a power of 2. The exponents are: (a) n^2. (b) $n(n-1)$. (c) $n(n-1)/2$. (d) $n(n+1)/2$.

CHAPTER SIX

1. 79,833,600.

2. Respectively, 210, 98.

3. 23,562.

4. Respectively, 48, 21.

5. 2036.

6. (a) $\{n\}_2$. (b) $\{n\}_3$.

7. (a) 7!. (b) 192. (c) 12.

8. (a) m if $n = 2m$ or $n = 2\mathrm{m} + 1$. (b) Either 61 or 62. (c) 41, 42, or 43. (d) Maximum at either $n - 1$ or n.

9. (a) *Hint:* f and g are each one-to-one and onto. (b) $f_f(x) = f(i)$ where $f(i) = x$. (c) For some i, $g(i) \neq \phi(i)$, hence if $(i, x) \in f$ then $g_f(x) \neq \phi_f(x)$.

10. (a) $g_f = \{(a, d), (b, c), (c, e), (d, b), (e, a)\}$.
(b) $g = \{(1, c), (2, b), (3, d), (4, e), (5, a)\}$.

11. Since $Q_0 \cap Q_1 = \varnothing$, power Q = power Q_0 + power Q_1.

12. Proof is given in the Appendix.

13. *Hint:* Replace x and y each by 1.

14. (a) 4. (b) 10. (c) 210. (d) 15.

15. (a) $n - 1$. (b) $\{n\}_2$. (c) $\{\{n\}_2\}_4$. (d) $3 \cdot \{n\}_4$.

16. (a) 63. (b) 624. (c) $(n+1)^n - 1$.

CHAPTER SEVEN

1. $20!/(8!5!5!2!)$.

2. k^2.

3. (a) $(k)_r$. (b) $\{k\}_r$.

4. (a) 4200. (b) Respectively, 1, 3, 6, 10, 13, 14. (c) In set language, if two sets of power j are different then the complements are also different. (d) Respectively, 13, 10, 6, 3, 1.

5. (a) 1540. (b) 1140.

6. Respectively, 1, 2, 3, 3, 2, 1.

7. 35.

8. Straight flushes are included. (a) 5148. (b) 2,598,960. (c) 56 and 4.

9. The functions are:
(a) $\{(1, 1), (2, 2), (3, 2), (4, 2), (5, 4)\}$.
(b) $\{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2)\}$.
(c) $\{(1, 1), (2, 1), (3, 1), (4, 4), (5, 4)\}$.

10. (a) {2, 4, 5, 6, 9}. (b) {3, 4, 5, 6, 7}. (c) {2, 3, 4, 8, 9}.

11. Let $A = A_1 \cup A_2 \cup \cdots \cup A_k$. Assume r does not exceed the least power of the sets A_i. Let the elements in a given combination of power r be numbered $1, 2, \ldots, r$. The least possible α value is 2 and the greatest is $r + k$. The number of combinations of r elements from the set $\{2, 3, 4, \ldots, r + k\}$ is $\{r + k - 1\}_r$.

12. 6.

13. $2\{m + n - 1\}_m - n$.

14. *Hint:* Consider the subsets of power m of $\{2, 3, 4, \ldots, m + n\}$ no two elements of which are consecutive integers.

CHAPTER EIGHT

1. (a) (1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 4), (1, 2, 3), (1, 5), (2, 2, 2), (2, 4), (3, 3), (6).
(b) (1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 2), (1, 1, 1, 1, 3), (1, 1, 1, 2, 2), (1, 1, 1, 4), (1, 1, 2, 3), (1, 1, 5), (1, 2, 2, 2), (1, 2, 4), (1, 3, 3), (1, 6), (2, 2, 3), (2, 5), (3, 4), (7).
(c) There are 22.

2. Respectively, 30, 42.

3. (a) (1, 1, 1, 1, 5), (1, 1, 1, 2, 4), (1, 1, 1, 3, 3), (1, 1, 2, 2, 2), (1, 2, 2, 2, 2).
(b) (1, 1, 1, 1, 6), (1, 1, 1, 2, 5), (1, 1, 1, 3, 4), (1, 1, 2, 2, 4), (1, 1, 2, 3, 3), (1, 2, 2, 2, 3), (2, 2, 2, 2, 2).
(c) (1, 1, 1, 1, 7), (1, 1, 1, 2, 6), (1, 1, 1, 3, 5), (1, 1, 1, 4, 4), (1, 1, 2, 2, 5), (1, 1, 2, 3, 4), (1, 1, 3, 3, 3), (1, 2, 2, 2, 4), (1, 2, 2, 3, 3), (2, 2, 2, 2, 3).
(d) There are 13.

4. (a) (1, 1, 1, 6), (1, 1, 2, 5), (1, 1, 3, 4), (1, 2, 2, 4), (1, 2, 3, 3), (2, 2, 2, 3).
(b) Same as (b) of Exercise 3.
(c) (1, 1, 1, 1, 1, 6), (1, 1, 1, 1, 2, 5), (1, 1, 1, 1, 3, 4), (1, 1, 1, 2, 2, 4), (1, 1, 1, 2, 3, 3), (1, 1, 2, 2, 2, 3), (1, 2, 2, 2, 2, 2).
(d) (1, 1, 1, 1, 1, 1, 6), (1, 1, 1, 1, 1, 2, 5), (1, 1, 1, 1, 1, 3, 4), (1, 1, 1, 1, 2, 2, 4), (1, 1, 1, 1, 2, 3, 3), (1, 1, 1, 2, 2, 2, 3), (1, 1, 2, 2, 2, 2, 2).

5. $P_k(n)$.

6. (a) 2. (b) 3. (c) $P(n)$.

7. (a) 2. (b) 5.

8. and **9.** Multiply the expression and compare your analysis with the result.

10. Replace each x_i by 1.

11. Write down n "1's," and consider the $n - 1$ spaces between them. Choose any $m - 1$ of these spaces and insert a separator. Each such configuration corresponds to a function of the desired kind. Hence, $\{n - 1\}_{m-1}$.

12. 2^{n-1}.

CHAPTER NINE

1. 9.1: Theorem 8, Chapter 5; 9.2: Theorem 10, Chapter 5; 9.3: Theorem 4, Chapter 6; 9.4: Theorem 4, Chapter 7; 9.5: Theorem 2, Chapter 8.

2. No distribution possible if $k < n$, one distribution if $k \geqq n$.

3. Same as in Exercise 2.

4. Assume $n \leqq k$. Index the set of elements $\{a_1, a_2, \ldots, a_n\} = A$. Let $A_1 \cup A_2 \cdots \cup A_q$ be a partition of A (see Exercise 5, Chapter 8). For each i let the elements in A_i be in the same cell. Each such partition of A corresponds to a distribution of this kind, and there are 2 for $n = 2$, 5 for $n = 3$, 18 for $n = 4$.

5. Assume $n \leqq 2k$. Let $A = \{a_1, a_2, \ldots, a_n\}$ be the set of elements and $C = \{c_1, c_2, \ldots, c_k\}$ be the set of cells. Consider a matrix of n rows and k columns.

$$\begin{matrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1k} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{n1} & \theta_{n2} & \cdots & \theta_{nk} \end{matrix}$$

in which

$$\theta_{ij} = \begin{cases} 1 & \text{if } a_i \text{ is in cell } c_j, \\ 0 & \text{if } a_i \text{ is not in cell } c_j. \end{cases}$$

Each such matrix corresponds to a distribution of this kind if $\sum_{j=1}^{k} \theta_{ij} = 1$ for each i and if $\sum_{i=1}^{n} \theta_{ij} \leqq 2$ for each j. If the number of distributions

for a given n and k is $N(n, k)$, then

$$N(2, 2) = 4; \quad N(2, 3) = 9; \quad N(2, k) = k^2;$$

$$N(3, 2) = 6; \quad N(3, 3) = 24; \quad N(3, k) = (k + 1)_3.$$

6. Assume $n \leqq 2k$. Consider the function $f: I_k \to \{0, 1, 2\}$ where $(x, y) \in f$ means that cell c_x contains y elements. If $\sum_{x=1}^{k} f(x) = n$, then f represents a distribution of this kind. In the notation of the preceding exercise,

$$N(2, 2) = 3; \quad N(3, 2) = 2; \quad N(4, 2) = 1;$$

$$N(2, 3) = 6; \quad N(3, 3) = 7; \quad N(4, 3) = 6;$$

$$N(5, 3) = 3; \quad N(6, 3) = 1.$$

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